Classical and Quantum Nonlocal Gravity

经典与量子非局域引力

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Abstract

摘要

This chapter of the Handbook of Quantum Gravity aims to illustrate how nonlocality can be implemented in field theories, as well as the manner it solves fundamental difficulties of gravitational theories. We review Stelle's quadratic gravity, which achieves multiplicative renormalizability successfully to remove quantum divergences by modifying the Einstein's action but at the price of breaking the unitarity of the theory and introducing Ostrogradski's ghosts. Utilizing nonlocal operators, one is able not only to make the theory renormalizable but also to get rid of these ghost modes that arise from higher derivatives. We start this analysis by reviewing the classical scalar field theory and highlighting how to deal with this new kind of nonlocal operators. Subsequently, we generalize these results to classical nonlocal gravity and, via the equations of motion, we derive significant results about the stable vacuum solutions of the theory. Furthermore, we discuss the way nonlocality could potentially solve the singularity problem of Einstein's gravity. In the final part, we examine how nonlocality induced by exponential and asymptotically polynomial form factors preserves unitarity and improves the renormalizability of the theory.

本章出自《量子引力手册》，旨在阐述非局域性如何在场论中实现，以及非局域性如何解决引力理论的基础性难题。我们综述了斯蒂尔二次引力: 该理论通过修正爱因斯坦作用量，成功实现了可乘重整性以消除量子发散，但代价是破坏了理论的幺正性，并引入了奥斯特罗格拉德斯基鬼。利用非局域算符，我们不仅可以让理论保持重整性，还能摆脱这些由高阶导数产生的鬼模式。我们从经典标量场论出发展开分析，阐明了如何处理这类新型非局域算符。随后，我们将上述结论推广到经典非局域引力，并通过运动方程推导得到了该理论稳定真空解的重要结论。此外，我们还讨论了非局域性解决爱因斯坦引力奇点问题的潜在途径。最后，我们研究了由指数和渐近多项式形状因子诱导的非局域性如何保持幺正性并改善理论的重整性。

Keywords

关键词

Models of quantum gravity - Nonlocality - Field theory - Classical theories of gravity - Renormalization and regularization

量子引力模型 - 非定域性 - 场论 - 经典引力理论 - 重整化与正则化

Conventions

约定

Throughout this chapter, we will employ the following conventions:

本章通篇采用以下约定:

- Minkowski spacetime with mostly plus signature, i.e., $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1\,, +1)_{\mu\nu}$.
  - 闵氏时空采用号差大部分为正的约定，即 $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1\,, +1)_{\mu\nu}$ 。
- Einstein summation convention, i.e., $\sum_i a_i b^i \equiv a_i b^i$.
  - 爱因斯坦求和约定，即 $\sum_i a_i b^i \equiv a_i b^i$ 。
- Greek indices run over the 4 spacetime dimensions whereas Roman indices run only over the 3 spatial dimensions.
  - 希腊指标遍历全部 4 个时空维度，拉丁指标仅遍历 3 个空间维度。
- We work in natural units, i.e., $\hbar = 1$ and $c = 1$.
  - 我们采用自然单位制，即 $\hbar = 1$ 和 $c = 1$ 。
- The Laplace-Beltrami operator or d’Alembertian $\Box$ is defined as $\Box = \nabla^\mu \nabla_\mu$.
  - 拉普拉斯-贝尔特拉米算子，即达朗贝尔算子 $\Box$，定义为 $\Box = \nabla^\mu \nabla_\mu$ 。
- The gravitational coupling κ is defined as $\kappa^2 = 8\pi G$, where G is Newton’s constant.
  - 引力耦合常数 κ 定义为 $\kappa^2 = 8\pi G$ ，其中 G 为牛顿引力常数。
- $A_{[\mu} B_{\nu]}$ denotes anti-symmetrization, i.e., $A_{[\mu} B_{\nu]} = \frac{1}{2}\left(A_\mu B_\nu - A_\nu B_\mu\right)$.

- $A_{[\mu}B_{\nu]}$ 表示反对称化，即 $A_{[\mu}B_{\nu]} = \frac{1}{2}\left(A_\mu B_\nu - A_\nu B_\mu\right)$。

- $A_{(\mu}B_{\upsilon)}$ denotes symmetrization, i.e., $A_{(\mu}B_{\upsilon)} = \frac{1}{2}\left(A_\mu B_\upsilon + A_\upsilon B_\mu\right)$.

- $A_{(\mu}B_{\upsilon)}$ 表示对称化，即 $A_{(\mu}B_{\upsilon)} = \frac{1}{2}\left(A_\mu B_\upsilon + A_\upsilon B_\mu\right)$。

- The Riemann tensor is defined as $R^{\rho}_{\mu\nu\sigma} = 2\partial_{[\nu}\Gamma^{\rho}_{\sigma]\mu} + 2\Gamma^{\lambda}_{\mu[\nu}\Gamma^{\rho}_{\sigma]\lambda}$.

- 黎曼张量定义为 $R^{\rho}_{\mu\nu\sigma} = 2\partial_{[\nu}\Gamma^{\rho}_{\sigma]\mu} + 2\Gamma^{\lambda}_{\mu[\nu}\Gamma^{\rho}_{\sigma]\lambda}$。

- The Ricci tensor is defined as $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$.

- 里奇张量定义为 $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$。

- The Ricci scalar is defined as $R = g^{\mu\upsilon}R_{\mu\upsilon}$.

- 里奇标量定义为 $R = g^{\mu\upsilon}R_{\mu\upsilon}$。

- The Einstein tensor is defined as $G_{\mu\upsilon} = R_{\mu\upsilon} - \frac{1}{2}Rg_{\mu\upsilon}$.

- 爱因斯坦张量定义为 $G_{\mu\upsilon} = R_{\mu\upsilon} - \frac{1}{2}Rg_{\mu\upsilon}$。

- To simplify the index notation, we will refer to the curvature operators $R_{\mu\upsilon\rho\sigma}$, $R_{\mu\upsilon}$, and R as $\mathcal{R}$.

- 为简化指标记号，我们将曲率算子 $R_{\mu\upsilon\rho\sigma}$、$R_{\mu\upsilon}$ 和 R 简记为 $\mathcal{R}$。

Introduction

引言

In the early twentieth century, there occurred two important revolutions in physics. On the one hand, Albert Einstein published what he called the theory of general relativity (GR), in which he was able to describe all the observable gravitational effects through the action

二十世纪初, 物理学领域发生了两次重大革命。一方面, 阿尔伯特·爱因斯坦发表了广义相对论 (GR), 他通过作用量描述了所有可观测的引力效应

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R. \tag{1}$$

For a local observer, GR recovered special relativity, which Einstein published in 1905 and, together with the posterior geometrical formalism of Hermann Minkowski, introduced the notion of spacetime. This theory was also able to account for new phenomena unexplained by Newtonian gravity, such as the perihelion precession of Mercury, the gravitational time dilation, and the time delay of the light travelling near a massive object [1]. However, along with this novel theory there appeared mathematical predictions for new astronomical objects in the universe called black holes, which opened new mysteries that GR was unable to solve, such as the singularity problem, that is, the impossibility of the theory to make predictions in some regions of spacetime where the curvature diverges.

> 对于局域观测者而言，广义相对论还原了爱因斯坦 1905 年发表的狭义相对论，狭义相对论与赫尔曼·闵可夫斯基后来提出的几何形式化共同引入了时空的概念。广义相对论还可以解释牛顿引力无法解释的新现象，例如水星近日点进动、引力时间膨胀，以及大质量天体附近传播的光的时间延迟 [1]。然而，伴随着这一全新理论，数学上预言了宇宙中一种被称为黑洞的新型天体，也带来了广义相对论无法解决的新谜题，例如奇点问题——即该理论无法在曲率发散的某些时空区域做出预言。

On the other hand, coevally to this theory of gravity, it was discovered that quantum mechanics could successfully describe the microscopic world. From the 1920s on, prominent physicists such as Dirac, Heisenberg, Pauli and later Yukawa, Feynman, Schwinger, Dyson, and others developed the formalism of quantum field theory (QFT) within the second quantization prescription and the path-integral formalism. Soon it was shown the great precision this new theory could achieve to compute observable quantities in particle scattering processes.

> 另一方面，与引力理论同期，人们发现量子力学可以成功描述微观世界。自 20 世纪 20 年代起，狄拉克、海森堡、泡利等杰出物理学家，以及后来的汤川秀树、费曼、施温格、戴森等人，在二次量子化方案和路径积分形式体系的框架下发展出了量子场论 (QFT) 的形式化体系。很快人们就发现，这套新理论在计算粒子散射过程中的可观测物理量时能够达到极高精度。

After the success of the theory of quantum electrodynamics (QED) in the 1940s and 1950s and having been established a solid classical theory of gravity, there appeared several attempts trying to quantize the gravitational theory in the same way done for the other fundamental forces of Nature. However, the complications arising from the very definition of this rank-2 symmetric field theory and its direct correspondence with geometry impeded its realization for decades. During the second half of the twentieth century and until nowadays, some approaches to this problem of quantum gravity were proposed, such as supergravity, string theory, loop quantum gravity, and others. To various degrees, each of them is well-formulated within their particular framework, and each of them should give a certain level of theoretical success as well as shortcomings, such as the lack of an empirical confirmation.

> 在 20 世纪 40 到 50 年代量子电动力学 (QED) 取得成功，且引力的经典理论已经建立之后，学界出现了诸多尝试，试图仿照对自然界其他基本力的处理方法对引力理论进行量子化。然而，这个二阶对称场理论从定义本身就带来了诸多复杂问题，加上它与几何的直接对应，导致量子化的尝试数十年间都未能成功。自二十世纪下半叶至今，人们已经提出了量子引力问题的多种研究方案，例如超引力、弦理论、圈量子引力等等。在各自的特定框架下，这些方案都在不同程度上得到了良好的表述，也都在理论上取得了一定成果，同时也存在缺陷，例如都缺乏经验观测证实。

In this chapter, we focus on the so-called nonlocal quantum gravity (NLQG) [2,3] (Not to be confused with the homonymous proposal of [4,5]. Unfortunately, there is no commonly established nomenclature classifying different nonlocal gravitational theories.). Although initially considered in the late 1980s, it has been a hot topic during the last twenty years thanks to several indications that renormalizability and unitarity are indeed achieved. However, before we jump into the technical machinery of NLQG, we will shortly review gravity with higher-order derivatives to show what the main problems of local models are.

本章我们聚焦于所谓的非局域量子引力 (NLQG)[2,3](请勿与文献 [4,5] 中的同名研究方案混淆，遗憾的是，目前尚未有通用命名规则对不同的非局域引力理论进行分类)。尽管这一方向最早在 20 世纪 80 年代末就被提出，近二十年来它成为了研究热点，因为多方面迹象表明该理论确实满足可重整性和幺正性。不过，在深入讨论 NLQG 的技术框架之前，我们会先简要回顾高阶导数引力，说明局域模型存在的主要问题。

Higher-Derivative Gravity

高导数引力

Higher-derivative theories constitute a generalization of GR in which one inserts extra curvature operators in the Einstein's action (1):

高导数理论是广义相对论的推广，它在爱因斯坦作用量 (1) 中引入了额外曲率算符:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R + f\left(R, R_{\mu\nu}, R_{\mu\nu\rho\sigma}\right) \right]. \tag{2}$$

In the attempt to develop a theory of quantum gravity, many types of models emerged, among which we highlight the following two. Although these theories improved our knowledge about quantum gravity, the presence of extra propagating degrees of freedom will become a problem both classically and at the quantum level.

在尝试发展量子引力理论的过程中，涌现出了多种模型，其中我们重点介绍以下两种。尽管这些理论增进了我们对量子引力的认知，但额外传播自由度的存在无论在经典层面还是量子层面都会成为一个问题。

$f(R)$ Gravity

$f(R)$ 引力

$f(R)$ gravity is a class of theories characterized by a function of the Ricci scalar R . Proposed in the 1970s by Buchdahl [6] and, in parallel, by Breizman, Gorovich, and Sokolov [7], these models have the action

$f(R)$ 引力是一类由里奇标量 R 的函数定义的高阶引力理论。该类模型由布赫达尔 [6] 以及布莱兹曼、戈罗维奇和索科洛夫 [7] 分别于 1970 年代提出，其作用量为

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R), \tag{3}$$

and one recovers Einstein’s gravity when $f(R) = R$. A particular element of this class of theories has gained reputation in cosmology. Starobinsky inflation [8] is described by the Lagrangian density

当 $f(R) = R$ 时该理论退化为爱因斯坦引力，这类理论中的一个特例在宇宙学领域得到了广泛关注。斯塔罗宾斯基暴胀 [8] 由如下拉格朗日密度描述

$$\mathcal{L} = R + \frac{1}{6m^2}R^2, \tag{4}$$

where m is a mass scale. Curvature produces an accelerated expansion at early times that has important applications in primordial cosmology [9].

其中 m 是一个质量标度。曲率在极早期产生加速膨胀，这在原初宇宙学中具有重要应用 [9]。

Stelle's Gravity

施泰勒引力

Stelle [10] developed a theory of gravity with a quadratic action:

施泰勒 [10] 提出了一个带有二次作用量的引力理论:

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right). \tag{5}$$

This theory has interesting properties, for instance, that the additional coupling constants

该理论拥有诸多有趣性质，例如额外的耦合常数

$$\frac{\alpha_i}{2\kappa^2}$$

are dimensionless, whose implications in renormalization will be crucial as it will be explained in section "Power-Counting Renormalizability".

是无量纲的，正如我们会在“幂次计数可重整性”小节中说明的，这一点对重整化而言至关重要。

Note that the Riemann-Riemann term can be removed in four dimensions since the Gauss-Bonnet action

注意到，在四维空间中黎曼-黎曼项可以消去，因为高斯-博内作用量

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\mathcal{L}_{\mathrm{GB}},\ \mathcal{L}_{\mathrm{GB}} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{6}$$

is proportional to the Euler number $\chi(\mathcal{M})$ that characterizes the topology of the manifold $\mathcal{M}$. The equations of motion derived from this action are trivial provided there is no topology change and no boundary terms. We will use this topological result to write the Riemann-Riemann term as a combination of the Ricci-Ricci scalar and the Ricci-Ricci tensor term.

正比于表征流形 $\mathcal{M}$ 拓扑性质的欧拉数 $\chi(\mathcal{M})$。在不存在拓扑变化与边界项的情况下，从该作用量推导得到的运动方程是平凡的。我们会利用这一拓扑结论，将黎曼-黎曼项改写为里奇标量与里奇张量项的组合。

Unstable Modes and Ostrogradski's Theorem

不稳定模式与奥斯特罗格拉德斯基定理

In 1850, Ostrogradski [11] showed that higher-derivative classical theories have instabilities. This instability translates to a spontaneous decay of the vacuum, which results in the inability to consider such theories as physical ones to describe our universe. To see how this result arises, one can follow the Hamiltonian approach [12] for two prototypical cases: theories with two derivatives and with four derivatives in the kinetic term.

1850 年，奥斯特罗格拉德斯基 [11] 证明了高阶导数经典理论存在不稳定性。这种不稳定性会引发真空自发衰变，导致这类理论无法作为描述我们宇宙的物理理论。要理解该结论的推导过程，可以沿着哈密顿方法 [12] 研究两个典型案例: 动能项含二阶导数、动能项含四阶导数的理论。

Two-Derivative Theories

二阶导数理论

Considering a Lagrangian depending only on x and $\dot{x}$, one has that the Euler-Lagrange equations of this system are given by

考虑一个仅依赖于 x 和 $\dot{x}$ 的拉格朗日量，该系统的欧拉-拉格朗日方程可表示为

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \Rightarrow \ddot{x} = F(x, \dot{x}) \Rightarrow x(t) = x(t, x_0, \dot{x}_0), \tag{7}$$

and assuming non-degeneracy, i.e., $\partial^2 L/\partial \dot{x}^2 \neq 0$, one is able to write the Hamiltonian of the system through a Legendre transform. The Hamiltonian is conserved in the absence of explicit time dependence. Furthermore, one finds that this Hamiltonian is positive definite and therefore bounded from below.

假设非退化性，即 $\partial^2 L/\partial \dot{x}^2 \neq 0$ ，我们就可以通过勒让德变换得到系统的哈密顿量。当拉格朗日量不显含时间时，哈密顿量是守恒量。此外，可以发现该哈密顿量是正定的，因此它有下界。

In the context of field theory, we can reproduce these results using the simplest scalar field theory, i.e., through the Klein-Gordon equation

在场论语境下，我们可以通过最简单的标量场理论，也就是克莱因-戈登方程，得到上述结果:

$$S = -\int d^4x \left(\frac{1}{2}\partial^\mu \phi \partial_\mu \phi + \frac{1}{2}m^2\phi^2\right) \Rightarrow (\Box - m^2)\phi = 0. \tag{8}$$

For this case, one is able to construct the Hamiltonian density as

这种情况下，我们可以构造出哈密顿密度如下:

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}\left(\partial_i\phi\partial^i\phi\right) + \frac{1}{2}m^2\phi^2 \geq 0, \tag{9}$$

where $\Pi = \dot{\phi}$ is the momentum associated with the field ϕ . In (9), we see that $\mathcal{H}$ is bounded from below so that the theory is stable as will be explained shortly.

其中 $\Pi = \dot{\phi}$ 是场 ϕ 对应的动量。我们可以从式 (9) 中看到 $\mathcal{H}$ 有下界，因此该理论是稳定的，我们很快就会对此进行说明。

Four-Derivative Theories

四导数理论

Similarly, we can consider a Lagrangian depending only on $x, \dot{x}$, and $\ddot{x}$. In this case, one has that the Euler-Lagrange equations are

同理，我们可以考虑一个仅依赖于 $x, \dot{x}$ 和 $\ddot{x}$ 的拉格朗日量。在此情况下，欧拉-拉格朗日方程为

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} = 0 \Rightarrow \ddddot{x} = F\left(x, \dot{x}, \ddot{x}, \dddot{x}\right) \Rightarrow x\left(t\right) = x\left(t, x_0, \dot{x}_0, \ddot{x}_0, \dddot{x}_0\right), \tag{10}$$

and assuming non-degeneracy, one is able to consistently write a Hamiltonian which is also constant in time if L is time independent. However, the main difference here is that this new Hamiltonian is no longer bounded from below, as one can see analysing the following four-derivative field theory:

并且假设非退化性，当 L 不依赖时间时，我们可以一致地写出一个不随时间变化的哈密顿量。但此处的核心区别在于，这个新的哈密顿量不再有下界，通过分析下面这个四导数场论就可以看出:

$$S = \int d^4x \left[\frac{1}{2}\phi\left(\Box - \alpha\Box^2\right)\phi - \frac{1}{2}m^2\phi^2\right] \Rightarrow \left(\Box - \alpha\Box^2 - m^2\right)\phi = 0. \tag{11}$$

From this, one can build the associated Hamiltonian density through a Legendre transform to see that

由此，我们可以通过勒让德变换得到对应的哈密顿密度，结果为

$$\mathcal{H} \propto \Pi_1\dot{\phi} + \mathcal{O}\left[\Pi_2^2, \left(\nabla\phi\right)^2, \phi^2\right], \tag{12}$$

where $\Pi_1 = \dot{\phi} - \alpha\dddot{\phi}$ and $\Pi_2 = \alpha\ddot{\phi}$. In this case, the linearity of $\mathcal{H}$ on ϕ and Π_1 prevents us to declare the Hamiltonian to be positive definite. Although this Hamiltonian is conserved, the linear term may take any value, even a negative one. Therefore, one concludes that $\mathcal{H}$ is unbounded from below.

其中 $\Pi_1 = \dot{\phi} - \alpha\dddot{\phi}$ 和 $\Pi_2 = \alpha\ddot{\phi}$ 。在此情形下，$\mathcal{H}$ 对 ϕ 和 Π_1 的线性性导致我们无法断言哈密顿量是正定的。尽管该哈密顿量是守恒量，但其线性项可以取任意值，甚至负值。因此我们可以得出结论: $\mathcal{H}$ 无下界。

In the massless case $m = 0$ of (11), the bare propagator in momentum space is given by

对于 (11) 的无质量情形 $m = 0$ ，动量空间中的裸传播子为

$$G\left(k^2\right) = -\frac{1}{k^2\left(1 + \alpha k^2\right)} = -\frac{1}{k^2} + \frac{\alpha}{1 + \alpha k^2}. \tag{13}$$

Identifying the poles of the propagator as the spectrum of the theory, we see that the presence of $\Box^2$ at the Lagrangian level introduces an extra pole in the propagator, leading to an additional massive mode of mass $m = 1/\sqrt{\alpha}$, provided $\alpha > 0$. However, the different sign in the propagator indicates that this massive mode corresponds to a ghost [13]. Furthermore, considering a homogeneous field $\phi(t)$, (11) becomes

将传播子的极点识别为理论的能谱后我们可以发现，拉格朗日量层面的 $\Box^2$ 会给传播子引入一个额外极点，在满足 $\alpha > 0$ 的条件下，该极点对应一个质量为 $m = 1/\sqrt{\alpha}$ 的额外有质量模式。而传播子中符号的差异表明，这个有质量模式对应一个鬼场 [13]。此外，考虑均匀场 $\phi(t)$ 时，式 (11) 变为

$$\alpha\ddddot{\phi} - \ddot{\phi} = 0, \tag{14}$$

whose real solution for $\alpha > 0$ is

其 $\alpha > 0$ 的实解为

$$\phi(t) = A\cosh\frac{t}{\sqrt{\alpha}} + B\sinh\frac{t}{\sqrt{\alpha}} + Ct + D, \tag{15}$$

where A, B, C , and D are integration constants and this solution is clearly a non-oscillating and unbounded function.

其中 A, B, C 和 D 是积分常数，该解显然是一个无振荡的无界函数。

The main consequence is that, already at the classical level, this system is unstable since the vacuum state can decay into excited states of particles and antiparticles contributing positively and negatively, respectively, to $\mathcal{H}$ [12]. This decay is not only possible but favoured from the entropy point of view [12] and, therefore, one concludes that this kind of theory is irreconcilable with the observed universe because our ground state would be plagued by highly excited modes that do not decouple at some high energy, as it would happen in a stable theory. Moreover, this continuum decay is uncontrollable unlike to what happens, for instance, in QED. In addition, at the quantum level, one would obtain again a Hamiltonian that could be negative valued, being able to excite indefinitely a quantum state with unbounded energies and creating pairs of particle-antiparticle spontaneously that, in turn, would decay into higher-energy pairs of particles leading to a continuum decay of the vacuum state.

这带来的核心结论是，即使在经典层面该系统也是不稳定的: 真空可以衰变为粒子和反粒子的激发态，二者分别对 $\mathcal{H}$ 产生正贡献和负贡献 [12]。这种衰变不仅是可能的，从熵的角度来看它还是自发发生的 [12]，因此我们可以得出结论: 这类理论与我们观测到的宇宙无法相容，因为我们的基态会被高激发模式困扰，这些模式不会像稳定理论中那样在高能标退耦。此外，这种连续衰变是不可控的，这和量子电动力学中的情况完全不同。而在量子层面，得到的哈密顿量同样可以取负值，量子态可以被无限激发，获得无界的能量，还会自发产生正反粒子对，这些粒子对又会衰变为更高能的粒子对，最终导致真空发生连续衰变。

Ghost Modes

鬼模

These unstable additional modes are called ghost modes since, as we have argued, the addition of higher-derivative terms in the Lagrangian comes at the price of producing extra degrees of freedom that are not observed in Nature. The kinetic term of these ghost fields appear in the Lagrangian with the wrong sign. At the quantum level, this translates into negative-norm states, leading to a violation of unitarity.

这些不稳定的附加模式被称为鬼模，正如我们已经说明的，在拉格朗日量中加入高阶导数项的代价是会产生自然界中并未观测到的额外自由度。这些鬼场的动能项在拉格朗日量中符号错误。在量子层面，这对应产生负范数态，进而导致幺正性被破坏。

Nonlocal Classical Scalar Field Theory

非局部经典标量场论

Before we focus on nonlocal gravity, we study a nonlocal scalar field theory in order to illustrate how fundamental nonlocality manifests itself in the classical dynamics. We introduce the concepts of form factors and kernel that will characterize the kind of nonlocality we have in our theory. We also show the problem of the initial conditions that these theories have to face and finally we expose how to solve it through the so-called diffusion method.

在我们聚焦非局部引力之前，先研究非局部标量场论，以说明基础非局域性如何在经典动力学中体现。我们会引入形状因子和核的概念，二者是我们理论中非局域性类型的特征。我们也会说明这类理论必须面对的初始条件问题，最后阐述如何通过所谓扩散法解决该问题。

Motivation

研究动机

From the very beginning of the foundations of QFT, the main sectors of physical interest, such as QED, have been assumed to be local, in the sense that the fields only depend on one spacetime coordinate x^{μ} . This field theory, however, can be generalized to include nonlocality, in which some fields of the Lagrangian are evaluated at two different spacetime points. For instance, in the 1930s, Wataghin entertained the idea to introduce nonlocality to give a finite size to point-like particles [14], and to do so he considered operators that were highly suppressed at large energies. For historical reasons, these operators were called form factors, since their original purpose was to give 'size' or 'form' to dimensionless particles.

从量子场论基础建立之初，量子电动力学等主要研究方向就被默认是定域的，即场仅依赖于单个时空坐标 x^μ 。但该场论可以推广为包含非定域性的形式: 拉格朗日量中的部分场会在两个不同的时空点取值。例如，1930 年代，瓦京就提出了引入非定域性给类点粒子赋予有限大小的想法 [14]，为此他考虑了在高能区高度压低的算符。由于历史原因，这些算符被称为形状因子，因为它们最初的目的就是给无量纲粒子赋予“尺寸”或“形状”。

Although local field theories prevailed, mainly because of the great predictivity of the Standard Model, nonlocal QFT has been an area of research in the second half of the twentieth century and many authors have applied these ideas to quantum scalar fields, gauge fields, gravity, and cosmology.

尽管定域场论占据了主流——这主要得益于标准模型极强的预言能力——但在 20 世纪下半叶，非定域量子场论一直是研究领域，许多学者已经将这些思想应用于量子标量场、规范场、引力和宇宙学研究中。

In general, one distinguishes between two kinds of nonlocality: the nonlocality induced by a Lagrangian valued at a field that depends on different points on spacetime $\mathcal{L}\left[\phi\left(x,y\right)\right]$, or the nonlocality arising in a Lagrangian that depends on multiple fields evaluated at different spacetime points, i.e., $\mathcal{L}\left[\phi\left(x\right),\phi\left(y\right)\right]$. Whereas the former nonlocality appears in the formulation of standard QFT, e.g., in multi-point multi-tensor objects such as Green's functions, here we focus on the latter, which is present in many areas of theoretical physics such as noncommutative QFT [15], string field theory [16], effective field theories [17], and conformal field theory [18-20].

一般而言, 非局域性分为两类: 一类由依赖于时空不同点的场的拉格朗日量诱导产生 $\mathcal{L}\left[\phi\left(x,y\right)\right]$, 另一类非局域性产生于依赖于多个不同时空点处场取值的拉格朗日量，即 $\mathcal{L}\left[\phi\left(x\right),\phi\left(y\right)\right]$ 。前者非局域性出现在标准量子场论的表述中，例如格林函数这类多点多张量对象，本文我们聚焦于后者，这类非局域性广泛存在于理论物理的诸多领域，如非对易量子场论 [15]、弦场论 [16]、有效场论 [17] 以及共形场论 [18-20]。

Nonlocality

非局域性

In the most common local field theory, the action and the equations of motion of the system are given by

在最常见的局域场论中，系统的作用量和运动方程由下式给出

$$S=\int d^4x\left[\frac{1}{2}\phi\left(x\right)\Box\phi\left(x\right)-V\left(\phi\right)\right]\Rightarrow\ \Box\phi\left(x\right)-\frac{dV}{d\phi\left(x\right)}=0. \tag{16}$$

One may introduce fundamental nonlocality by considering a more general function $\rightarrow\ \gamma\left(\Box\right)$. More precisely, we consider not a finite sum of higher-derivative terms, but an infinite sum of these operators in the following form:

我们可以通过考虑更一般的函数 → $\gamma(\Box)$ 引入基本非局域性。更准确地说，我们不考虑有限项高阶导数项之和，而是以下列形式写出这些算符的无穷和:

$$\gamma(\Box) = \sum_{n=0}^{\infty} c_n \Box^n \tag{17}$$

so that the previous action (16) becomes

因此先前的作用量 (16) 变为

$$S = \int d^4x \left[\frac{1}{2}\phi(x)\gamma(\Box)\phi(x) - V(\phi)\right]. \tag{18}$$

The equations of motion of this theory can be obtained via

该理论的运动方程可以通过下式得到

$$\frac{\delta S}{\delta\phi(x)} = 0 \;\Rightarrow\; \gamma(\Box)\phi(x) - \frac{dV}{d\phi(x)} = 0. \tag{19}$$

The connection of this class of theories with nonlocality, although at first sight not apparent, can be seen through the following algebraic manipulation:

这类理论与非局域性的关联乍看并不明显，但通过下列代数变换就能看出:

$$\begin{aligned}
\gamma(\Box)\phi(x) &= \int d^4k\gamma\left(-k^2\right)\delta^{(4)}(k^\mu - i\nabla^\mu)\phi(x) \\
&= \int d^4k \int d^4y e^{-iy\cdot k} K(y)\,\delta^{(4)}(k^\mu - i\nabla^\mu)\phi(x) \\
&= \int d^4y K(y)\, e^{y\cdot\nabla}\phi(x) \\
&= \int d^4y K(y)\,\phi(x+y) \\
&= \int d^4y K(y-x)\,\phi(y),
\end{aligned} \tag{20}$$

where in the last equality we have applied a change of variables and we have used the Fourier transform along the way. We see that the action $\gamma(\Box)$ on $\phi(x)$ applies a delocalization of the field, so that the kinetic term

最后一个等式中我们进行了变量替换，并在此过程中使用了傅里叶变换。可以看出，作用在 $\phi(x)$ 上的作用量 $\gamma(\Box)$ 对场进行了离域化，因此动能项

$$\phi(x)\gamma(\Box)\phi(x)=\int d^4y\phi(x)K(y-x)\phi(y), \tag{21}$$

relates two scalar fields evaluated at two different spacetime points via the delocalization kernel $K(y-x)$. Thus, and somewhat surprisingly, any kinetic term can be formally regarded as a nonlocal interaction. We call $\gamma(\Box)$ form factor and this is precisely what Wataghin was interested in. From (21), one sees that

> 通过离域化核 $K(y-x)$ 关联了两个不同时空点上的两个标量场。因此，有点令人惊讶的是，任何动能项都可以在形式上被视为非局域相互作用。我们称 $\gamma(\Box)$ 为形状因子，这正是沃塔金所研究的内容。从式 (21) 可以看出

$$K(x)=\gamma(\Box_x)\delta^{(4)}(x)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\gamma\left(-k^2\right). \tag{22}$$

The form of $\gamma(\Box)$ determines the particle spectrum of the free theory via the dispersion relation

> $\gamma(\Box)$ 的形式通过色散关系决定了自由理论的粒子谱

$$\gamma\left(-k^2\right)=0. \tag{23}$$

For instance, one could consider the usual local scalar field theory (16) with $V=0$, and in this case $\gamma\left(-k^2\right)=-k^2=0$, that is the dispersion relation for the massless scalar field theory. The integral representation of the form factor also works even for local higher-derivative theories. Then, the result is special and the kernel K is just a sum of Dirac deltas and their derivatives of finite order. In general, for nonlocal theories K is a smooth continuous function of the difference of the coordinates $x-y$ (because of translational invariance).

> 例如，我们可以考虑带有 $V=0$ 的常规局域标量场论 (16)，此时 $\gamma\left(-k^2\right)=-k^2=0$ 就是无质量标量场论的色散关系。即使对局域高阶导数理论，形状因子的积分表示也成立。此时结果很特殊，核 K 只是狄拉克 δ 函数及其有限阶导数的和。一般而言，对于非局域理论，K 是坐标差 $x-y$ 的光滑连续函数 (这源于平移不变性)。

In general, from a kernel $K(x,y)$ defined at two different spacetime points and composing the operator $\int dy\phi(x)K(x,y)\phi(y)$, it is possible to obtain a kinetic term $\phi(x)\gamma(\Box)\phi(x)$ only in very special situations. From this observation, another classification of nonlocality distinguishes three types.

> 一般来说，从定义在两个不同时空点上、构成算符 $\int dy\phi(x)K(x,y)\phi(y)$ 的核 $K(x,y)$ 出发，只有在非常特殊的情况下才能得到动能项 $\phi(x)\gamma(\Box)\phi(x)$。基于这一观察，非局域性的另一种分类将其分为三类。

- Weak nonlocality, when the kinetic term is an analytic function of the Laplace-Beltrami operator. Then, the form factor $\gamma(z)$ admits a regular Taylor series expansion around the infrared (IR) point $\Box=z=0$.

> - 弱非局域性: 动能项是拉普拉斯-贝尔特拉米算符的解析函数。此时形状因子 $\gamma(z)$ 可以在红外 (IR) 点 $\Box=z=0$ 附近做正则泰勒展开。

- Strong nonlocality, with a form factor of the type $\Box^{-1}$ or other inverse powers $\Box^{-n}$ singular at $z = \Box = 0$.

> - 强非局域性: 形状因子为 $\Box^{-1}$ 类型，或是其他在 $z = \Box = 0$ 处奇异的逆幂次 $\Box^{-n}$ 。

- Very strong nonlocality, when the kinetic term is made of an integral kernel $K(x, y)$ not convertible into a derivative operator $\gamma(\Box)$. (In this case, the function $\gamma(\Box)$ can be written only formally and it really does not exist due to convergence problems of the inverse Fourier transform from (22).)

> - 极强非局域性: 动能项由无法转化为微分算符 $\gamma(\Box)$ 的积分核 $K(x, y)$ 构成。(这种情况下，函数 $\gamma(\Box)$ 只能形式上写出，由于逆傅里叶变换 (22) 的收敛性问题，它实际上并不存在。)

Inside the class of weakly nonlocal form factors, we can distinguish those which are exponential in powers of the $\Box$ [14, 16, 21] , those which are asymptotically polynomial in the ultraviolet (UV) regime along the real axis [2, 22, 23] , and also fractional powers of the $\Box$ [24-26]. NLQG is characterized by weak nonlocality and asymptotically polynomial form factors [2, 22, 23] , although one can formulate quantum gravity also with fractional operators [27, 28]. Higher-derivative local theories are when the form factor is a finite polynomial and its exponents are positive integer numbers [29, 30] .

> 在弱非局域形状因子类别中，我们可以区分出在 $\Box$ [14, 16, 21] 幂次上为指数形式的形状因子、在实轴 [2, 22, 23] 的紫外 (UV) 区渐近多项式的形状因子，还有 $\Box$ 的分数次幂 [24-26]。非局域量子引力 (NLQG) 的特征是弱非局域性和渐近多项式形状因子 [2, 22, 23]，不过我们也可以用分数算符构造量子引力 [27, 28]。高阶导数局域理论的形状因子是有限多项式，其指数为正整数 [29, 30] 。

Representations of Form Factors

> ## 形状因子的表示

In general, one may define a nonlocal operator in two different ways or representations: the integral representation or the series representation. It is expected that weakly nonlocal form factors admit both representations, although they can give very different results when applied to certain seed functions. As an example, apply the cosmological $\Box$ operator with Hubble parameter $H \propto 1/t$ on a power law t^n : The series representation of $e^{\Box}$ fails to converge, while the integral one gives a finite answer [31].

> 一般而言，非局域算符可以通过两种不同的方式或表示定义: 积分表示与级数表示。一般认为弱非局域形状因子同时具备这两种表示，不过将二者应用于某些原函数时会得到截然不同的结果。举例来说，将哈勃参数为 $H \propto 1/t$ 的宇宙学 $\Box$ 算符作用于幂律 t^n : $e^{\Box}$ 的级数表示无法收敛，而积分表示可以给出有限结果 [31]。

Series Representation

> ## 级数表示

Already introduced in Eq. (17), this is the most common representation and it is based on the decomposition of the nonlocal form factor $\gamma(\Box)$ as an infinite sum of the Laplace-Beltrami operator. Analogously to what happens when trying to Taylor expand a function around a value which is not contained in the domain of the function, this representation is only valid for form factors $\gamma(z)$ that are regular at $z = 0$. For instance, the form factor $\Box^{-1}$ does not admit a series representation without regularization. Finally, we point out that this representation has the same form and structure in any spacetime background.

该表示已在式 (17) 中引入，是最常用的表示方法，它基于将非形状因子 $\gamma(\Box)$ 分解为拉普拉斯-贝尔特拉米算子的无穷级数。与在函数定义域外的点对函数做泰勒展开的情况类似，该表示仅适用于在 $z = 0$ 处正则的形状因子 $\gamma(z)$ 。例如，形状因子 $\Box^{-1}$ 在没有正则化的情况下无法使用级数表示。最后我们指出，该表示的形式与结构在任意时空背景下都保持不变。

Integral Representation

积分表示

The integral representation has been introduced in (20) in the scalar case using Minkowski spacetime as the background of the theory, with the kernel function, which is defined in (22). This representation is only valid in flat spacetime in Cartesian coordinates, since in general the Laplace-Beltrami operator is accompanied by terms dependent on the connection of the manifold. In this sense, we say that the decomposition in terms of the exponentials $\exp(\pm ik \cdot x)$ is no longer useful for such representation since they are no longer eigenfunctions of $\Box$.

在标量情形下，积分表示已于式 (20) 中引入，以闵氏时空作为理论背景，核函数由式 (22) 定义。该表示仅在笛卡尔坐标下的平直时空中有效，因为一般情况下拉普拉斯-贝尔特拉米算子会带有依赖于流形联络的项。就此而言，基于指数函数 $\exp(\pm ik \cdot x)$ 的分解对该表示不再适用，因为指数函数不再是 $\Box$ 的本征函数。

Therefore, when constructing explicit solutions of the nonlocal gravitational theory, first of all, one needs to build the integral representation of the nonlocal form factor out of the two eigenstates $\mathcal{B}_l$ of the Laplace-Beltrami operator in curved spacetime following this recipe [31]:

因此，构造非局部引力理论的显式解时，首先需要按照下述方法 [31]，从弯曲时空中拉普拉斯-贝尔特拉米算子的两个本征态 $\mathcal{B}_l$ 出发，构建非局部形状因子的积分表示:

1. Find the two eigenstates of the Laplace-Beltrami operator by solving the second-order differential equation

1. 通过求解二阶微分方程得到拉普拉斯-贝尔特拉米算子的两个本征态

$$\left(\Box + k^2\right) \mathcal{B}_l(k, x) = 0, \; l = 1, 2, \tag{24}$$

where k is either real or purely imaginary.

其中 k 为实数或纯虚数。

2. Write the field as a linear superposition of the eigenstates $\mathcal{B}_l$. For the scalar case we may define two integral transformations of the field as

2. 将场写为本征态 $\mathcal{B}_l$ 的线性叠加。对于标量情形，我们可以将场的两个积分变换定义为

$$\overline{\phi}_l(k) = \int d^4x \mathcal{B}_l(k,x) g_l(x) \phi(x), \tag{25}$$

with g_l being certain weights. Thus, the scalar field may be written as

其中 g_l 为确定权重。因此，标量场可写为

$$\phi(x) = \int d^4k \left[c_1 \mathcal{B}_1(k,x) \overline{\phi}_1(k) + c_2 \mathcal{B}_2(k,x) \overline{\phi}_2(k)\right], \tag{26}$$

where $c_{1,2} \in \mathbb{C}$ are constants and different for each particular field.

其中 $c_{1,2} \in \mathbb{C}$ 为常数，对每个特定场各不相同。

3. Write the nonlocal form factor acting on the scalar field as

3. 将作用在标量场上的非局部形状因子写为

$$\gamma(\Box)\phi(x) = c_1 \overline{\phi}_1(x) + c_2 \overline{\phi}_2(x), \tag{27}$$

where

其中

$$\overline{\phi}_l(x) = \int d^4k \mathcal{B}_l(k,x) \gamma\left(-k^2\right) \overline{\phi}_l(k). \tag{28}$$

Form Factors

形状因子

We are interested in making our quantum theory renormalizable but, as one could imagine, not all form factors will be viable for this purpose. In this section, we introduce form factors of interest in quantum gravity, and since the energy dimension of $\Box$ is $[\Box] = 2$, we will have to introduce a characteristic length in the theory, denoted by l_* . We will make explicit the momentum dependence of these form factors in the UV and how this behaviour for large momenta affects the renormalization of the theory.

我们希望让我们的量子理论可重整化，但不难想象，并非所有形状因子都能满足这一要求。本节我们将介绍量子引力中受关注的形状因子，且由于 $\Box$ 的能量量纲为 $[\Box] = 2$，我们需要在理论中引入特征长度，记为 l_*。我们将明确说明这些形状因子在紫外区的动量依赖关系，以及大动量下的这种行为如何影响理论的重整化。

Exponential Form Factor

指数形状因子

This form factor can be written as

该形状因子可以写为

$$\gamma(\Box) = \left(\Box - m^2\right) e^{-l_*^2 \Box}, \tag{29}$$

and the bare propagator (the inverse of γ) in momentum space is given by

而动量空间中的裸传播子 (即 γ 的逆) 由下式给出

$$G\left(k^2\right) = -\frac{e^{-l_*^2 k^2}}{k^2 + m^2}. \tag{30}$$

This expression is enlightening because one sees that it has only one real pole at $k^2 = -m^2$, so that the exponential form factor, unlike what happens with higher-derivative terms in (13), does not introduce additional modes into the spectrum of the theory.

这一表述很有启发性，因为可以看到它仅在 $k^2 = -m^2$ 处存在一个实极点，因此指数形状因子并不会像式 (13) 中的高阶导数项那样，给理论谱引入额外的模式。

One can also study the action of exponential form factors on some well-behaved functions that tend to zero very fast when their argument is sent to infinity (UV limit). One such example of the probe function is, in the one-dimensional case, the Gaussian function $f(x) = \exp\left[A(x-x_0)^2 + B\right]$, where $A < 0, B$, and x_0 are constants; this example can be easily generalized to the higher-dimensional case. The result of the action of the spacetime operator $\exp\left(a\partial_x^2 + b\partial_x + c\right)$ on f is, provided $1 - 4aA > 0$,

我们还可以研究指数形状因子对一类性质良好的函数的作用，这类函数当自变量趋于无穷大 (即紫外极限) 时会极快地趋近于零。一维情况下这类试探函数的一个典型例子是高斯函数 $f(x) = \exp\left[A(x-x_0)^2 + B\right]$，其中 $A < 0, B$ 和 x_0 都是常数；该例子可以很容易推广到高维情形。若满足条件 $1 - 4aA > 0$，时空算符 $\exp\left(a\partial_x^2 + b\partial_x + c\right)$ 作用于 f 的结果为，

$$\exp\left(a\partial_x^2 + b\partial_x + c\right) f(x) = \frac{\exp\left\{-\frac{A\left[-4a(B+c)+b^2-2b(x-x_0)+(x-x_0)^2\right]+B+c}{4aA-1}\right\}}{\sqrt{1-4aA}},$$

(31)

which is still a Gaussian function suppressed at spatial infinity. We see that the action of the exponential form factor translates into the transformation of the coefficients characterizing the original Gaussian:

结果仍是一个在空间无穷远处被压制的高斯函数。可见指数形状因子的作用等价于变换原高斯函数的特征系数:

$$A' = \frac{A}{4aA-1}, \; x_0' = x_0 + b, \; B' = -\frac{1}{2}\ln(1-4aA) + B + c. \tag{32}$$

Hence one can conclude that these are simple operations on the parabola defining the Gaussian, like shifting its vertex horizontally (with the b parameter), vertically (with the c parameter), and changing the shape or opening thereof (with the a parameter).

因此可以得出结论，这些都是定义高斯函数的抛物线上的简单操作: 比如借助 b 参数水平移动顶点，借助 c 参数垂直移动顶点， 以及借助 a 参数改变抛物线的形状或开口程度。

Therefore, by this example one unambiguously understands the action of nonlocal exponential form factors of basic one-dimensional differential operators on highly suppressed probe functions as just changing the shape and positions of these functions without modifying their asymptotic fall-off properties. For a general function with different infinite asymptotics, the action of the form factor may not be so well-defined and may possess some ambiguities.

因此通过这个例子可以明确得到: 一维基本微分算符的非局部指数形状因子，对高度压制探测函数的作用仅仅是改变这些函数的形状和位置，不会修改它们的渐近衰减性质。对于具有不同无穷渐近行为的一般函数，形状因子的作用可能没有这么良好的定义，还可能存在一些歧义。

More General Form Factors

更通用的形状因子

In NLQG we are interested mainly in four nonlocal operators, called Wataghin, Krasnikov, Kuz'min, and Tomboulis form factors. Here we write their definitions and their properties in the UV. We forewarn the reader that, while all these form factors work well for a scalar theory, only the last two (asymptotically polynomial) are eventually under full control in quantum gravity, since in the presence of gauge or diffeomorphism invariance there are residual divergence in loop diagrams when using the first two form factors (exponential). All of them can be written as

在非局部量子引力 (NLQG) 中，我们主要研究四种非局部算符，即瓦塔金 (Wataghin)、克拉斯尼科夫 (Krasnikov) 库兹明 (Kuz'min) 和通布利斯 (Tomboulis) 形状因子。本文给出它们的定义及其在紫外区的性质。我们提前提醒读者: 尽管所有这些形状因子都适用于标量理论，但最终只有后两种 (渐近多项式型) 能在量子引力中得到完全控制，因为使用前两种 (指数型) 形状因子时，若存在规范或微分同胚不变性，圈图中会残留发散。所有形状因子都可以写成如下形式

$$\gamma(\Box) = \frac{e^{\mathrm{H}(\Box)} - 1}{\Box}, \tag{33}$$

where $\frac{1}{\Box}$ stands for the inverse Laplace-Beltrami operator (always absorbed by the leading $O(\Box)$ term in the numerator; hence there is no strong nonlocality here) and the function $\mathrm{H}(\Box)$ will be different for each case. Equation (33) has the same purpose as in (30) not to introduce new poles, except the already existing one of the massless spin-2 graviton from the Einstein-Hilbert term in the case of quantum gravity (If this assumption is not used, then one can even fancy models without any physical degree of freedom where the kinetic term has the simple form $\exp \mathrm{H}(\Box)$. p -adic models have this structure [32].).

其中 $\frac{1}{\Box}$ 代表逆拉普拉斯-贝尔特拉米算子 (它总会被分子中的领头项 $O(\Box)$ 吸收；因此此处不存在强非局域性)，函数 $\mathrm{H}(\Box)$ 在每种情形下各不相同。式 (33) 的作用和式 (30) 一致，即不引入新极点，仅保留量子引力情形下爱因斯坦-希尔伯特项中原本就存在的无质量自旋 2 引力子极点 (若不采用该假设，甚至可以构造出没有任何物理自由度的模型，其动力学项具有简单形式 $\exp \mathrm{H}(\Box)$ 。p 进模型就具有这种结构 [32]。)。

Wataghin/Minimal Form Factor

沃塔金/极小形状因子

Characterized by $\mathrm{H}_{\mathrm{Wat}}(\Box) = -l_*^2\Box$, it is the form factor that has been used the most due its simplicity, although nowadays the emphasis in NLQG has been displaced to asymptotically polynomial form factors. Its expression is given by

它以 $\mathrm{H}_{\mathrm{Wat}}(\Box) = -l_*^2\Box$ 为特征，是目前使用最广泛的形状因子，原因在于其构造简单，不过现如今 NLQG 的研究重点已经转移到了渐近多项式形状因子上。其表达式为

$$\gamma(\Box) = \frac{e^{-l_*^2\Box} - 1}{\Box}. \tag{34}$$

Krasnikov Form Factor

克拉斯尼科夫形状因子

Defined in 1987 by Krasnikov [21], it is characterized by $\mathrm{H}_{\mathrm{Kras}}(\Box) = l_*^4\Box^2$:

由克拉斯尼科夫于 1987 年定义 [21]，其特征为 $\mathrm{H}_{\mathrm{Kras}}(\Box) = l_*^4\Box^2$:

$$\gamma(\Box) = \frac{e^{l_*^4\Box^2} - 1}{\Box}. \tag{35}$$

Kuz'min Form Factor

库兹明形状因子

Now we introduce a class of asymptotically polynomial form factors whose asymptotic behaviour is very different from the previously considered exponential type. We begin with form factors being asymptotically monomial, where the characteristic monomial is given by $m(z) = az^{n_{\text{deg}}}$ with a a constant and n_{deg} the degree of the monomial given by an integer number.

现在我们介绍一类渐近多项式形状因子，其渐近行为与之前讨论的指数型完全不同。我们从渐近单项式的形状因子开始，特征单项式由 $m(z) = az^{n_{\text{deg}}}$ 给出，其中 a 是常数，n_{deg} 是单项式的次数，为整数。

The function $\mathrm{H}(z)$ in these asymptotically monomial form factors must satisfy certain conditions.

这类渐近单项式形状因子中的函数 $\mathrm{H}(z)$ 必须满足若干条件。

1. It is real and positive along the real axis.

1. 它在实轴上是实正函数。

2. $\mathrm{H}(0) = 0$ such that $e^{\mathrm{H}(0)} = 1$.

2. $\mathrm{H}(0) = 0$ 满足 $e^{\mathrm{H}(0)} = 1$ 。

3. It grows no faster than a monomial of finite degree n_{deg} when one takes $|z| \to +\infty$ in a particular conical region $\mathcal{C}$ of the complex plane. This region is characterized by an angle θ with respect to the real axis. Thus we can write this condition mathematically as

3. 当 $|z| \to +\infty$ 处于复平面的特定锥形区域 $\mathcal{C}$ 时，它的增长速度不超过次数为 n_{deg} 的单项式。该区域相对于实轴的夹角为 θ ，因此我们可以将该条件数学表示为

$$e^{\mathrm{H}(z)} \overset{|z|\to\infty}{\simeq} |z|^{n_{\text{deg}}} \quad z \in \mathcal{C},\ -\theta < \arg(z) < \theta. \tag{36}$$

4. In the conical region

4. 在锥形区域内

$$\lim_{|z|\to\infty} \frac{e^{\mathrm{H}(z)} - |z|^{n_{\text{deg}}}}{|z|^{n_{\text{deg}}}} z^n = 0 \ \forall n \in \mathbb{N}. \tag{37}$$

The rationale behind asymptotically monomial form factors will be clearer in section ”Renormalization”, where the UV behaviour, dictated by the order n_{deg} of the polynomial, will be essential to achieve renormalizability.

渐近单项式形状因子的设计思路会在“重整化”一节变得更清晰，该节中由多项式次数 n_{deg} 决定的紫外行为对实现可重整性至关重要。

Kuz’min form factor is given by formula (33) with

库兹明形状因子由公式 (33) 给出，其中

$$\mathrm{H}(\Box) = \alpha\left[\ln(m(\Box)) + \Gamma[0, m(\Box)] + \gamma_{\mathrm{E}}\right] \equiv \mathrm{H}_{\mathrm{Kuz}}(\Box), \tag{38}$$

where γ_{E} is the Euler-Mascheroni constant, $m(\Box)$ is a real monomial ($a \in \mathbb{R}$) of degree $n_{\mathrm{deg}} \geq 1, \alpha$ is a real constant such that $\alpha \geq 3$ (in order to have one-loop super-renormalizability, i.e., perturbative UV divergences only at the one-loop level) and Γ is the upper incomplete gamma function defined as

> γ_{E} 是欧拉-马歇罗尼常数，$m(\Box)$ 是实单项式 ($a \in \mathbb{R}$)，次数为 $n_{\mathrm{deg}} \geq 1, \alpha$，$\alpha \geq 3$ 是满足条件 $\alpha \geq 3$ 的实常数 (为了得到一圈超可重整性，即仅在一圈阶存在紫外发散)，Γ 是上不完全伽马函数，定义为

$$\Gamma(0, z) = \int_z^{\infty} dx \frac{e^{-x}}{x}. \tag{39}$$

More precisely, Kuz'min [22] introduced this form factor in 1989 with $m(\Box) = -l_*^2\Box$, hence $n_{\mathrm{deg}} = 1$. This construction can be generalized to a polynomial asymptotic behaviour, as explained in section "General Asymptotically Polynomial Form Factor" below.

> 更准确地说，库兹明于 1989 年引入该形状因子时取 $m(\Box) = -l_*^2\Box$，因此得到 $n_{\mathrm{deg}} = 1$。如下文“一般渐近多项式形状因子”一节所述，该构造可以推广到多项式渐近行为。

Tomboulis Form Factor

> ## 通布利斯形状因子

This form factor is defined with the asymptotic polynomial $p = p(z)$ by

> 该形状因子由渐近多项式 $p = p(z)$ 定义如下

$$\mathrm{H}(\Box) = \frac{1}{2}\left\{\ln\left[p^2(\Box)\right] + \Gamma\left[0, p^2(\Box)\right] + \gamma_{\mathrm{E}}\right\} \equiv \mathrm{H}_{\mathrm{Tom}}(\Box), \tag{40}$$

whose domain includes the conical regions

> 其定义域包含以下锥形区域

$$\mathcal{C} = \{z \mid -\theta < \arg(z) < \theta \ \cup \ \pi - \theta < \arg(z) < \pi + \theta\},\ \theta = \frac{\pi}{4n_{\mathrm{deg}}},$$

(41)

depicted in Fig. 1 for the cases of monomials with degrees $n_{\mathrm{deg}} = 1, 2, 3$ respectively.

> 分别对应次数为 $n_{\mathrm{deg}} = 1, 2, 3$ 的单项式情形，如图 1 所示。

Asymptotically polynomial form factors are not everywhere asymptotic to a polynomial in the UV regime $|z| \gg 1$. They are such only inside the conical regions along the real axis. Outside these regions on the complex plane, they may have other higher-order asymptotics such as $\exp(\exp z)$, for example. Moreover, all these nontrivial entire functions have an essential singularity at the complex infinity $z = \infty_c$.

渐近多项式形状因子在紫外区 $|z| \gg 1$ 并非处处渐近于多项式，仅沿实轴的锥形区域内满足这一性质。在复平面的这些区域之外，它们可以拥有其他高阶渐近形式，例如 $\exp(\exp z)$ 。此外，所有这些非平凡整函数在复无穷远点 $z = \infty_c$ 都存在一个本性奇点。

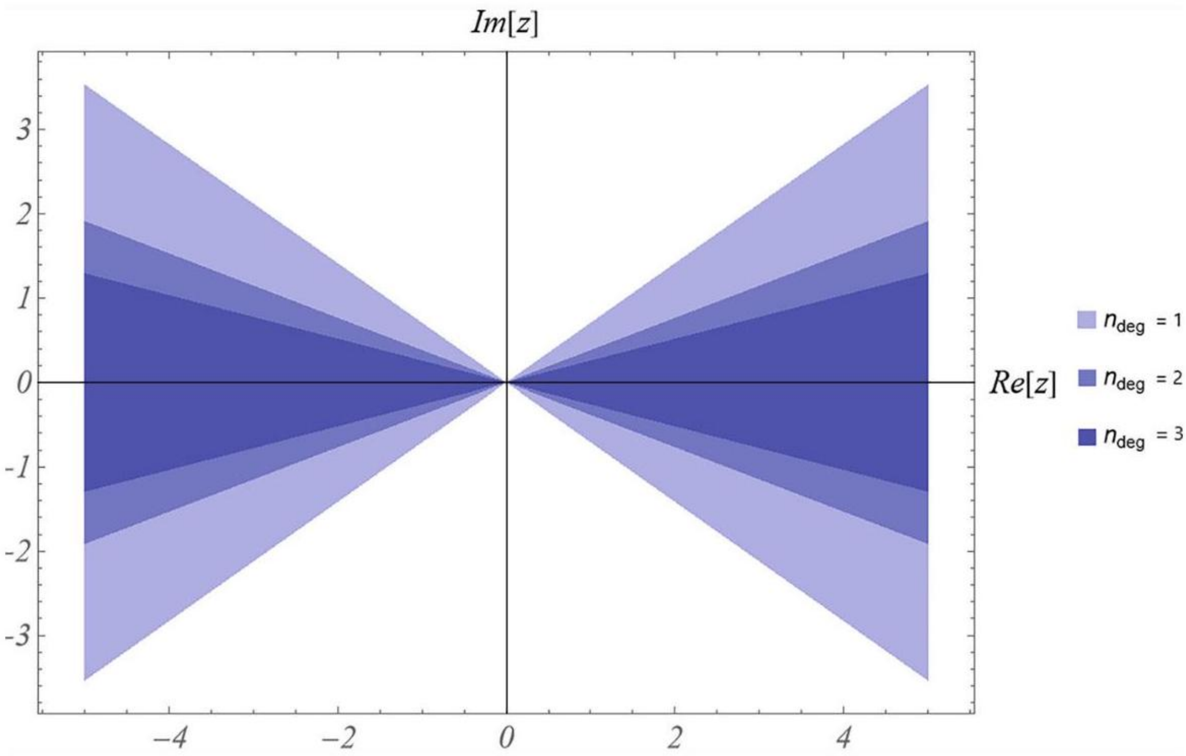


Fig. 1 Conical regions $\mathcal{C}$ of the Tomboulis form factor $\mathrm{H}_{\mathrm{Tom}}$ for $n_{\deg} = 1, 2, 3$

图 1 Tomboulis 形状因子 $\mathrm{H}_{\mathrm{Tom}}$ 对应 $n_{\deg} = 1, 2, 3$ 的锥形区域 $\mathcal{C}$

At any rate, form factors like Kuz'min and Tomboulis can be defined as entire functions also outside these conical regions and outside the essential singularity at infinity.

无论如何，库兹明和通布利斯这类形状因子都可以定义为整函数，其定义域也覆盖这些锥形区域之外、无穷远本性奇点之外的区域。

General Asymptotically Polynomial Form Factor

一般渐近多项式形状因子

We view an asymptotically polynomial form factor as a general analytic complex function of Fourier space momentum k_μ . Covariant form factors depend on $\square \rightarrow -k^2 = -g^{\mu\upsilon} k_\mu k_\upsilon : \gamma = \gamma(k^2)$. In Euclidean domain, $k^2 = \delta^{\mu\upsilon} k_\mu k_\upsilon$ and, calling $z = k^2$, we have

我们将渐近多项式形状因子视为傅里叶空间动量 k_μ 的一般解析复函数。协变形状因子依赖于 $\Box \to -k^2 = -g^{\mu\upsilon}k_\mu k_\upsilon$: $\gamma = \gamma(k^2)$ 。在欧几里得区域中，$k^2 = \delta^{\mu\upsilon}k_\mu k_\upsilon$ ，令 $z = k^2$ ，我们得到

$$\gamma = \gamma(z). \tag{42}$$

For $z \gg 1$ and in the asymptotic limits of $|z| \to +\infty$ in the conical region $\mathcal{C}$ along the real positive axis, we should have asymptotics to a polynomial $p = p(z)$ with a degree $n_{\text{deg}} \in \mathbb{N}$:

对于 $z \gg 1$ ，当 $|z| \to +\infty$ 处于沿正实轴的锥形区域 $\mathcal{C}$ 的渐近极限中时，我们得到渐近形式为多项式 $p = p(z)$ ，次数为 $n_{\text{deg}} \in \mathbb{N}$:

$$p(z) = a_{n_{\text{deg}}} z^{n_{\text{deg}}} + \sum_{i=0}^{n_{\text{deg}}-1} a_i z^i, \tag{43}$$

where $a_{n_{\text{deg}}} \neq 0$. Then, in the UV we have a higher-derivative local theory.

其中 $a_{n_{\text{deg}}} \neq 0$ 。此时，在紫外区域我们得到一个高阶导数局域理论。

In Euclidean signature, the UV regime is only when $k^2 \gg 1$ and this regime is responsible for the UV properties of the theory such as the structure of UV divergences, renormalizability, super-renormalizability, or UV-finiteness. In Lorentzian signature, the situation is more delicate. We have $k^2 = g^{\mu\upsilon}k_\mu k_\upsilon$ with metric signature $(-,+,+,+)$. The deep (physical) UV regime is then defined as two regimes, $k^2 \to +\infty$ ($|k^2| \gg 1$ on the real line) and $k^2 \to -\infty$ ($|k^2| \gg 1$ on the real line and negative value for the Lorentzian square k^2). However, a possible UV regime could also arise when $|\mathbf{k}| = |k^0| \to +\infty$, i.e., when we are on the light cone $k^2 = -(k^0)^2 + |\mathbf{k}|^2 = 0$. In such a special condition, the analysis of the UV behaviour of the form factor is carried out at the argument $z = k^2 = 0$. The UV divergences which could arise in such situation would not be preserving Lorentz symmetry, since the condition on the component $k^0 \gg 1$ is not imposed on the full Lorentz-invariant length of the four-vector k^μ .

在欧几里得号差下，紫外区域仅对应 $k^2 \gg 1$ ，该区域决定了理论的紫外性质，例如紫外发散的结构、可重整性、超可重整性或紫外有限性。在洛伦兹号差下，情况更为复杂。我们有 $k^2 = g^{\mu\upsilon}k_\mu k_\upsilon$ ，度规号差为 $(-,+,+,+)$ 。因此深度 (物理) 紫外区域分为两类: $k^2 \to +\infty$ ($|k^2| \gg 1$ (实线上) 和 $k^2 \to -\infty$ ($|k^2| \gg 1$ (实线上，且洛伦兹平方 k^2 为负值)。不过，当 $|\mathbf{k}| = |k^0| \to +\infty$ 即我们处于光锥 $k^2 = -(k^0)^2 + |\mathbf{k}|^2 = 0$ 上时，也可能存在紫外区域。在这种特殊条件下，形状因子的紫外行为需要在自变量 $z = k^2 = 0$ 处分析。这种情况下产生的紫外发散无法保持洛伦兹对称性，因为对分量 $k^0 \gg 1$ 的约束并非施加在四矢量 k^μ 的完整洛伦兹不变长度上。

For the sake of good renormalizability properties of the Lorentzian theory, one should require that the asymptotically polynomial behaviour holds in the two regimes $k^2 \to \pm\infty$ with the same polynomial and that the conical regions are symmetric with respect to the origin point. The asymptotics is such that, for $|z| \to +\infty$, we should have in the conical regions that $\gamma(z) \simeq p(z)$:

为了保证洛伦兹理论具备良好的可重整性，应当要求渐近多项式行为在两个区域 $k^2 \to \pm\infty$ 中对同一个多项式成立，且锥形区域关于原点对称。渐近行为满足: 对于 $|z| \to +\infty$ ，在锥形区域中应有 $\gamma(z) \simeq p(z)$:

$$\lim_{\substack{|z|\to+\infty \\ z\in\mathcal{C}}} \frac{\gamma(z)}{p(z)} = 1. \tag{44}$$

The difference should be suppressed as

差值应当满足如下压低关系

$$\lim_{\substack{|z|\to+\infty \\ z\in\mathcal{C}}} z^{\alpha}\left[\gamma(z) - p(z)\right] = 0, \tag{45}$$

for any $\alpha \in \mathbb{R}$, especially for $\alpha > 0$. Then, it is guaranteed that the analysis of perturbative UV divergences gives the same result in both regimes $k^2 \to \pm\infty$ and coincides with the analysis of infinities in local higher-derivative theory described by the polynomial $p(z)$.

对任意 $\alpha \in \mathbb{R}$ ，尤其是 $\alpha > 0$ 成立。由此可以保证，微扰紫外发散的分析在两个区域 $k^2 \to \pm\infty$ 中得到相同结果，并且与多项式 $p(z)$ 描述的局域高阶导数理论中的无穷大分析一致。

A more general condition for the natural order n of the form factor understood as the complex function is

对于作为复函数理解的形状因子，其自然阶 n 的一个更一般条件是

$$\lim_{\substack{|z|\to+\infty \\ z\in\mathcal{C}}} \frac{\ln\gamma}{\ln z} = n \tag{46}$$

so that the function here has the order n in the analysis of the asymptotics of complex entire functions. In this case, only the leading UV divergences could be captured correctly by a local higher-derivative theory with the monomial $a_n z^n$. However, even this general case and definition does not imply that

因此，在复整函数的渐近分析中，此处函数的阶为 n 。这种情况下，只有领头紫外发散能被带有单项式 $a_n z^n$ 的局域高阶导数理论正确捕获。但即便是这种一般情形和定义，也不意味着

$$\lim_{\substack{|z|\to+\infty \\ z\in\mathcal{C}}} \frac{\gamma}{a_n z^n} = 1 \tag{47}$$

The last equality holds for truly asymptotically polynomial form factors with degree $n_{\deg} = n$, while for general complex functions of order n it may be not satisfied.

最后一个等式对阶为 $n_{\deg} = n$ 的真渐近多项式形状因子成立，而对于阶为 n 的一般复函数，它不一定成立。

Summary of Form Factors

形状因子总结

We can summarize the UV properties of the previous form factors.

我们可以总结前文所述各形状因子的紫外性质。

1. Wataghin, Krasnikov, Kuz'min, and Tomboulis form factors diverge in the UV in Euclidean momentum space, i.e., $k_E^0 = ik^0$, as one can see in Fig. 2.

1. 瓦塔金、克拉斯尼科夫、库兹明和通布利斯形状因子在欧几里得动量空间的紫外区发散，即 $k_E^0 = ik^0$ ，如图 2 所示。

Therefore, since all of them blow up in the UV, the associated propagator, which is the inverse of the form factor, will be highly suppressed in the UV, thus facilitating the convergence of Feynman diagrams for the scalar case. (The case of gauge theories and gravitation is more complicated since also interaction vertices include form factors.) Furthermore, the huge growth of the form factors in Euclidean momentum space implies asymptotic freedom: since the kinetic term dominates in the UV when considering its running along the renormalization-group energy scale, interactions become negligible and the resulting theory is asymptotically free. We will come back to this point in section "Asymptotic Freedom". We also emphasize that these form factors are not UV-divergent on the light cone where the massless on-shell dispersion relation $k^2 = -(k^0)^2 + |\mathbf{k}|^2 = 0$ is satisfied, since $\mathrm{H}(0) = 0$. Hence there is a suppression of the propagator in the UV everywhere except on the light cone.

因此，由于所有这类形状因子都会在紫外区激增，其对应的传播子 (即形状因子的逆) 会在紫外区被强烈压低，从而促进标量情形下费曼图的收敛。(规范理论和引力的情形更复杂，因为相互作用顶点也包含形状因子。) 此外，形状因子在欧几里得动量空间的大幅增长意味着渐近自由: 当我们沿重整化群能标跑动分析时，动能项在紫外区占主导，相互作用变得可以忽略，因此得到的理论是渐近自由的。我们会在“渐近自由”一节回到这一问题。我们还要强调，这些形状因子在光锥上并不紫外发散——无质量在壳色散关系 $k^2 = -(k^0)^2 + |\mathbf{k}|^2 = 0$ 在光锥上成立，这是因为 $\mathrm{H}(0) = 0$ 。因此除光锥外，传播子在所有位置的紫外区都会被压低。

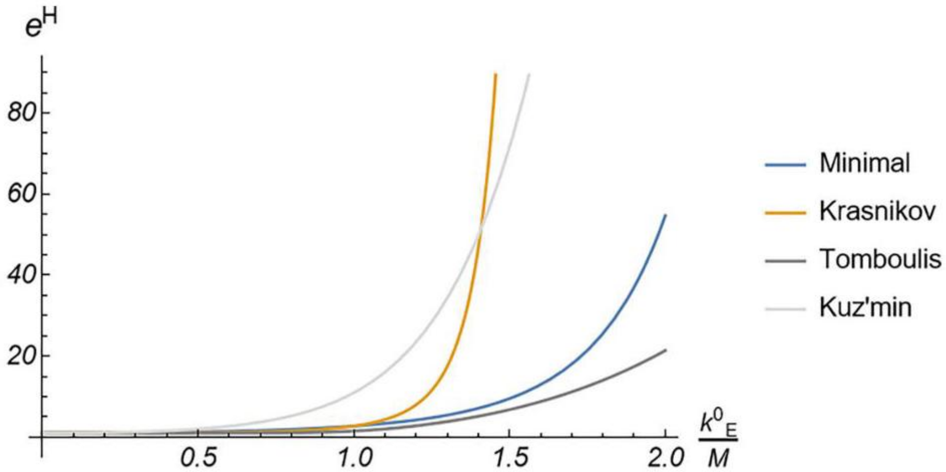


Fig. 2 exp (H) as a function of the Euclidean momentum k_E^0 , where we have defined the energy scale $M = 1/l_*$ and taken $\alpha = 3$ and $m(\Box) = -\Box$ for Kuz' min form factor and a monomial with $n_{\text{deg}} = 2$ for Tomboulis form factor

图 2 以欧几里得动量 k_E^0 为自变量的 exp (H)，图中我们定义了能标 $M = 1/l_*$，对库兹明形状因子取 $\alpha = 3$ 和 $m(\Box) = -\Box$，对通布利斯形状因子取满足 $n_{\text{deg}} = 2$ 的单项式

2. Their behaviour is different in Lorentzian momentum space. In particular we have that

2. 这类形状因子在洛伦兹动量空间的行为不同。具体来说，我们得到

$$\lim_{k^0\to\infty} \mathrm{H}\left(-k^2\right) = -\infty \quad \text{for Wataghin and Kuz' min form factors,} \tag{48}$$

and

且

$$\lim_{k^0\to\infty} \mathrm{H}\left(-k^2\right) = +\infty \quad \text{for Krasnikov and Tomboulis form factors,} \tag{49}$$

as one sees in Fig. 3. This difference occurs because the first group of form factors is sensitive to the sign of their argument $z = k^2$, while in the second group the dependence of the argument is always quadratic $\left(z^2 \text{ or } p^2(z)\right)$ and the limits $k^2 \to \pm\infty$ give the same result.

如图 3 所示。这种差异产生的原因是: 第一组形状因子对自身自变量 $z = k^2$ 的符号敏感，而第二组形状因子的自变量始终是二次依赖 $\left(z^2 \text{ or } p^2(z)\right)$，取极限 $k^2 \to \pm\infty$ 会得到相同结果。

Stability and Initial Conditions

稳定性与初始条件

We saw in section "Unstable Modes and Ostrogradski's Theorem" that an immediate consequence of adding higher-derivative terms to Einstein's theory (1) is the intrusion of extra degrees of freedom, some of which in the form of ghost fields. Through the use of the nonlocal form factors introduced in section "Form Factors", we commented on the possible amelioration of the divergences of the Feynman diagrams. However, since all the form factors that we consider can be expanded in a power series of the Laplace-Beltrami operator, we might wonder how this procedure ensures that they do not actually add an infinite number of extra propagating modes. Moreover, the absence of ghosts is clear from the absence of extra poles but, looking from the side of the infinite sum of powers of $\Box$, how is it possible that these manage to cancel any Ostrogradski instability?

我们在“不稳定模式与奥斯特罗格拉德斯基定理”一节中看到，在爱因斯坦引力理论 (1) 中引入高阶导数项的直接后果是引入了额外自由度，其中部分自由度以鬼场的形式存在。借助“形状因子”一节中介绍的非局域形状因子，我们讨论了改善费曼图发散的可能性。但由于我们研究的所有形状因子都可以展开为拉普拉斯-贝尔特拉米算子的幂级数，我们不禁要问，这一过程如何保证这些形状因子实际上不会引入无穷多个额外传播模式？此外，不存在额外极点就说明没有鬼，但从 $\Box$ 幂次无穷和的角度来看，这些项怎么可能成功抵消所有奥斯特罗格拉德斯基不稳定性呢？

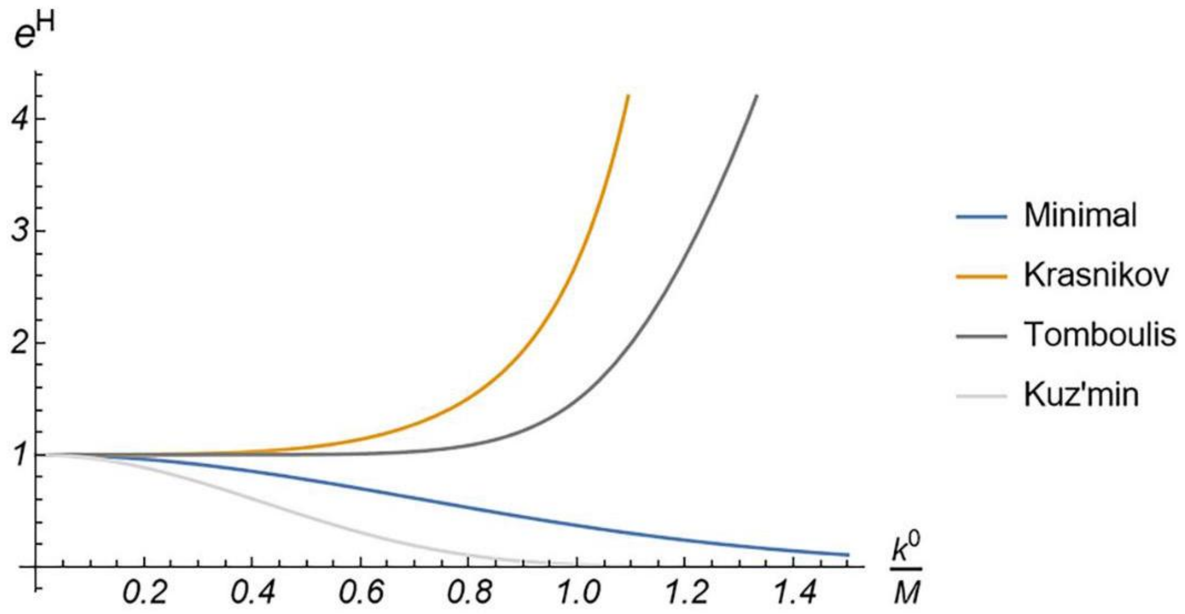


Fig. 3 exp (H) as a function of the Lorentzian momentum k^0 , where we have defined the energy scale $M = 1/l_*$ and taken $\alpha = 3$ and $m(\Box) = -\Box$ for Kuz' min form factor and a monomial with $n_{\text{deg}} = 2$ for Tomboulis form factor

图 3 exp (H) 作为洛伦兹动量 k^0 的函数，图中我们定义了能标 $M = 1/l_*$ ，对于库兹明形状因子取 $\alpha = 3$ 和 $m(\Box) = -\Box$ ，对于汤布利斯形状因子取单项式 $n_{\text{deg}} = 2$

In any classical theory with n derivatives in the Lagrangian, one needs to specify $2n$ initial conditions to solve uniquely the physical system. However, in the nonlocal case, the kinetic term contains an infinite number of derivatives, so that one might say that we need an input of an infinite number of initial conditions at $t = t_0$:

在拉格朗日量含 n 阶导数的任何经典理论中，需要指定 $2n$ 个初始条件才能唯一求解物理系统。但在非局域情形下，动能项包含无穷多阶导数，因此有人会认为我们需要在 $t = t_0$ 处给出无穷多个初始条件:

$$\phi^{(n)}(t_0, \mathbf{x}) \ \forall n \in \mathbb{N}. \tag{50}$$

However, these initial conditions are precisely what we need to construct the solution $\phi(t, \mathbf{x})$ as a power series, provided that the solution is real and analytic (which is different from the requirement that the solution be smooth and that it smoothly depend on initial data):

然而，只要解是实解析的 (这不同于解光滑且光滑依赖初始数据的要求)，这些初始条件恰好是我们将解 $\phi(t, \mathbf{x})$ 展开为幂级数所需要的全部条件:

$$\phi(t, \mathbf{x}) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(t_0, \mathbf{x})}{n!}(t - t_0)^n. \tag{51}$$

Thus, we face a paradox [33]. We need an infinite number of initial conditions to specify the solution $\phi(t, \mathbf{x})$ but, once we have them, we can directly construct our field via the expression (51) assuming reality and analyticity at $t = t_0$. To fully solve the problem of initial conditions, we should already know the solution!

因此我们面临一个悖论 [33]: 我们需要无穷多个初始条件才能确定解 $\phi(t, \mathbf{x})$ ，但一旦我们得到了这些条件，在假设解在 $t = t_0$ 处实且解析的前提下，我们就可以通过表达式 (51) 直接构造场。要完全解决初始条件问题，我们必须事先就知道解!

Diffusion Method

扩散方法

We can overcome this issue using the diffusion method, initially used in the context of string field theory [31, 34]. For simplicity, we illustrate the method for Wataghin form factor [35] but, after adaptations, it holds also for asymptotically polynomial form factors [36]. In essence, one promotes our four-dimensional field $\phi(t, \mathbf{x})$ into a field $\Phi(r, t, \mathbf{x})$ depending on a fictitious extra dimension r , with the constraint that $\Phi(r, t, \mathbf{x})$ obeys the diffusion equation on the r coordinate:

我们可以用扩散方法解决这个问题，该方法最初应用于弦场论背景 [31, 34]。为简化推导，我们针对 Wataghin 形状因子 [35] 演示该方法，但经过适配后，它也适用于渐近多项式形状因子 [36]。本质上，我们将四维场 $\phi(t, \mathbf{x})$ 推广为依赖于虚构额外维 r 的场 $\Phi(r, t, \mathbf{x})$ ，且要求 $\Phi(r, t, \mathbf{x})$ 满足关于 r 坐标的扩散方程:

$$(\partial_r - \Box)\,\Phi(r, t, \mathbf{x}) = 0. \tag{52}$$

Together with this extra dimension, one introduces an auxiliary field $\chi(r, t, \mathbf{x})$ to impose the diffusion equation on $\Phi(r, t, \mathbf{x})$ that, in turn, is constrained by $\chi = \Box\Phi$. In the above equation and everywhere below, $\Box$ is understood as the four-dimensional operator. Also, one can write an action with measure d^4xdr [35] but it is paramount to stress that the r -dependent system is never conceived as a physical five-dimensional extension of the original nonlocal theory. The extra direction is flat (it does not come with any warp factor) and there is no attempt to implement five-dimensional covariance, nor to study the localized system as a five-dimensional QFT to be made unitary or renormalizable. Rather, the four-dimensional system is regarded as living on a flat slice at some special value $r = \widetilde{\beta} r_*$ in this abstract ambient space, where $\widetilde{\beta} > 0$.

引入这个额外维的同时，我们还引入了辅助场 $\chi(r,t,\mathbf{x})$ ，以对受 $\chi=\Box\Phi$ 约束的 $\Phi(r,t,\mathbf{x})$ 施加扩散方程条件。在上式及下文所有推导中，$\Box$ 均为四维算符。此外，我们可以写出带测度 d^4xdr 的作用量 [35]，但必须强调，依赖 r 的体系绝非原非局部理论的物理五维推广。额外方向是平直的 (不存在翘曲因子)，我们也并不尝试构造五维协变，更不将这个局域化体系视作需要满足幺正性或可重整化的五维量子场论。相反，四维体系被视为存在于这个抽象空间中特定值 $r=\widetilde{\beta}r_*$ 处的平直切片上，此时 $\widetilde{\beta}>0$ 。

The motivation to constrain $\Phi(r,t,\mathbf{x})$ via a diffusion equation is that we can treat the exponential form factor as a translation in the extra dimension r :

我们通过扩散方程约束 $\Phi(r,t,\mathbf{x})$ 的动机是: 可以将指数形状因子处理为额外维 r 上的平移:

$$e^{-l_*^2\Box}\Phi(r,t,\mathbf{x})=e^{-l_*^2\partial_r}\Phi(r,t,\mathbf{x})=\Phi\left(r-l_*^2,t,\mathbf{x}\right), \tag{53}$$

and we recover the physical field configuration in four dimensions $\phi(t,\mathbf{x})$ for a specific value of the extra component r , i.e., at the particular slice location in r proportional to $r_*=l_*^2$. Note that $[r]=-2$, so that strictly speaking it is not on the same level as a coordinate. This choice is dictated to make the diffusion equation (52) parameter free, with diffusion coefficient equal to 1.

当额外分量 r 取特定值时，也就是在 r 中与 $r_*=l_*^2$ 成正比的特定切片位置，我们就能重新得到四维物理场构型 $\phi(t,\mathbf{x})$ 。请注意 $[r]=-2$ ，因此严格来说它并不与普通坐标处于同等地位。这个选择是为了让扩散方程 (52) 不含自由参数，扩散系数等于 1。

Procedure

步骤

To illustrate how this method works, we focus on the particular nonlocal system

为说明该方法的工作原理，我们聚焦于特定非局域系统

$$S_\phi=\int d^4x\mathcal{L},\ \mathcal{L}=\frac{1}{2}\phi\Box e^{-r_*\Box}\phi-V[\phi], \tag{54}$$

whose equation of motion is

其运动方程为

$$\Box e^{-r_*\Box}\phi-V'[\phi]=0. \tag{55}$$

In the definition of this model, we assume that the potential is a local function of the field $\phi=\phi(x)=\phi(t,x)$, without derivatives.

在定义该模型时，我们假设势是场 $\phi=\phi(x)=\phi(t,x)$ 的局域函数，不含导数。

First of all, we introduce two fields $\Phi(r,t,\mathbf{x})$ and $\chi(r,t,\mathbf{x})$ local in four-dimensional spacetime directions (i.e., their four-dimensional dynamics in x is local), while the nonlocality is completely transferred to the unphysical extra dimension r . By definition, we recover the physical dynamics of the nonlocal system when evaluating the field Φ at the special slice $r=\widetilde{\beta}r_*$ for a certain $\widetilde{\beta}>0$, such that $\Phi\left(\widetilde{\beta}r_*,t,\mathbf{x}\right)=\phi(t,\mathbf{x})$. Because of the locality of the dynamics of the field $\Phi(r,t,\mathbf{x})$ in the spacetime coordinates, we only require a finite number of initial conditions for the field $\phi(t,\mathbf{x})$.

首先，我们引入两个在四维时空方向上局域的场 $\Phi(r,t,\mathbf{x})$ 和 $\chi(r,t,\mathbf{x})$ (即它们在 x 中的四维动力学是局域的)，而非局域性被完全转移到非物理额外维 r 。根据定义，当我们在特殊切片 $r=\widetilde{\beta}r_*$ 上对场 Φ 求值，满足特定条件 $\widetilde{\beta}>0$ 使得 $\Phi\left(\widetilde{\beta}r_*,t,\mathbf{x}\right)=\phi(t,\mathbf{x})$ 时，即可还原非局域系统的物理动力学。由于场 $\Phi(r,t,\mathbf{x})$ 在时空坐标上的动力学是局域的，我们只需要为场 $\phi(t,\mathbf{x})$ 提供有限个初始条件。

One can build a suitable Lagrangian for the localized system [35]

我们可以为局域化系统构建一个合适的拉格朗日量 [35]

$$S[\chi,\Phi]=\int d^4xdr\left(\mathcal{L}_\chi+\mathcal{L}_\Phi\right), \tag{56}$$

where

其中

$$\mathcal{L}_\Phi=\frac{1}{2}\Phi(r,x)\Box\Phi(r-r_*,x)-V[\Phi(r,x)], \tag{57}$$

$$\mathcal{L}_\chi=\frac{1}{2}\int_0^{r_*}dq\chi(r-q,x)\left(\partial_q-\Box\right)\Phi(r+q-r_*,x), \tag{58}$$

and we have encapsulated the time component t and the spatial coordinates $\mathbf{x}$ into the label x . Notice that this action entails a local dynamics in the four-dimensional spacetime coordinates but nonlocal in the unphysical coordinate r . The equations of motion of this action are given by

我们已经将时间分量 t 和空间坐标 $\mathbf{x}$ 封装到记号 x 中。请注意，该作用量在四维时空坐标上是局域动力学，但在非物理坐标 r 上是非局域的。该作用量的运动方程为

$$\frac{\delta S}{\delta\chi}=0,\ \frac{\delta S}{\delta\Phi}=0, \tag{59}$$

leading to the expressions [35]

由此得到表达式 [35]

$$0=(\partial_r-\Box)\Phi(r,x), \tag{60}$$

$$0=(\partial_r-\Box)\chi(r,x), \tag{61}$$

$$0 = \frac{1}{2}\left[\Box\Phi\left(r - r_*, x\right) + \chi\left(r - r_*, x\right)\right]$$

$$+\frac{1}{2}\left[\Box\Phi\left(r + r_*, x\right) - \chi\left(r + r_*, x\right)\right] - V'\left[\Phi\left(r, x\right)\right], \tag{62}$$

where $V'\left[\Phi\left(r,x\right)\right] = dV/d\Phi\left(r,x\right)$.

其中 $V'\left[\Phi\left(r,x\right)\right] = dV/d\Phi\left(r,x\right)$ 。

Both fields Φ and χ follow a diffusion equation. The solutions to these equations of motion are not uniquely determined. However, we have a freedom left to impose an extra constraint

场 Φ 和 χ 都满足扩散方程。这些运动方程的解并不唯一，但我们可以额外施加一个约束条件

$$\Box\Phi\left(r, x\right) = \chi\left(r, x\right), \tag{63}$$

for any value of r , so that the auxiliary field χ freezes out at the physical slice and equation (62) becomes

对任意 r 成立，使得辅助场 χ 在物理切片上冻结，方程 (62) 变为

$$0 = \Box\Phi\left(r - r_*, x\right) - V'\left[\Phi\left(r, x\right)\right] = \Box e^{-r_*\partial_r}\Phi\left(r, x\right) - V'\left[\Phi\left(r, x\right)\right]$$

(64)

$$= \Box e^{-r_*\Box}\Phi\left(r, x\right) - V'\left[\Phi\left(r, x\right)\right].$$

Evaluating this expression at $r = \tilde{\beta}r_*$, i.e., the slice where $\Phi\left(\tilde{\beta}r_*, x\right) = \phi\left(x\right)$, one recovers the equation of motion of the physical nonlocal system (55).

在 $r = \tilde{\beta}r_*$ 即满足 $\Phi\left(\tilde{\beta}r_*, x\right) = \phi\left(x\right)$ 的切片上对该表达式求值，即可还原物理非局域系统 (55) 的运动方程。

Although the equivalence of both systems has been established for a particular theory (54) with an exponential form factor, this result can be generalized for other theories such as those with an exponential-polynomial form factor [35] or an asymptotically polynomial form factor [36].

尽管两个系统的等价性是针对具有指数形状因子的特定理论 (54) 证明的，但该结果可以推广到其他理论，例如具有指数-多项式形状因子 [35] 或渐近多项式形状因子 [36] 的理论。

Initial Conditions, Degrees of Freedom and Absence of Ghosts

初始条件、自由度与无鬼场

Given a nonlocal system, we can always write down a localized system whose solutions to the equations of motion at the physical slice $r = \tilde{\beta}r_*$ coincide with, or at least approximate, the ones of the original nonlocal system. The correspondence between both systems is not injective because we have required to impose a further constraint (63). Since the localized system (56) is second order in spacetime components, we only need two initial conditions for each field Φ and χ instead of an infinite number of them. In particular, we need $\Phi(r,t_0,\mathbf{x}), \dot{\Phi}(r,t_0,\mathbf{x}), \chi(r,t_0,\mathbf{x})$, and $\dot{\chi}(r,t_0,\mathbf{x})$ at all r and later evaluated at the special point with $r=\widetilde{\beta}r_*$. However, since we have imposed (63) by hand, we have that the fields χ and Φ are not independent and the initial conditions $\chi(r,t_0,\mathbf{x})$ and $\dot{\chi}(r,t_0,\mathbf{x})$ at all values of r are not independent from the ones of Φ , so that we end up needing only two initial conditions for the field $\Phi(r,x)$ to specify the solution of nonlocal dynamics.

给定一个非局域系统，我们总能构造一个局域化系统，其运动方程在物理切片 $r=\widetilde{\beta}r_*$ 上的解与原非局域系统的解一致，或至少对原解做了近似。两个系统间的对应不是单射，因为我们需要额外施加一个约束 (63)。由于局域化系统 (56) 在时空分量上是二阶的，每个场 Φ 和 χ 仅需要两个初始条件，而非无穷多个初始条件。具体而言，我们需要在所有 r 处给出 $\Phi(r,t_0,\mathbf{x}), \dot{\Phi}(r,t_0,\mathbf{x}), \chi(r,t_0,\mathbf{x})$ 和 $\dot{\chi}(r,t_0,\mathbf{x})$ ，之后再在满足 $r=\widetilde{\beta}r_*$ 的特殊点求值。但由于我们手动施加了约束 (63)，场 χ 和 Φ 并不独立，且所有 r 处的初始条件 $\chi(r,t_0,\mathbf{x})$ 和 $\dot{\chi}(r,t_0,\mathbf{x})$ 也不独立于 Φ 的初始条件，因此最终我们仅需要为场 $\Phi(r,x)$ 提供两个初始条件，就能确定非局域动力学的解。

Once we have established the equivalence between the nonlocal system and one slice of the localized system, we may proceed to count the degrees of freedom of the theory. Using the Hamiltonian formalism [35], one may show that the field $\chi(r,x)$ is a ghost mode and the associated Hamiltonian is unbounded from below in the ’five-dimensional’ system. However, in the physical slice $r=\widetilde{\beta}r_*$, this field disappears from the spectrum since its propagation is constrained by (63). Therefore, the absence of this ghost mode in the nonlocal four-dimensional system leaves us with only one propagating degree of freedom $\phi(x)$ that only requires two initial conditions $\phi(t_0,\mathbf{x})$ and $\dot{\phi}(t_0,\mathbf{x})$, plus the knowledge of the corresponding solution in the local system at $r_*=0$ (see below).

一旦我们确立了非局域系统与局域化系统一个切片之间的等价性，就可以继续计算该理论的自由度。利用哈密顿形式主义 [35] 可以证明，在这个“五维”系统中，场 $\chi(r,x)$ 是鬼模，对应的哈密顿下无界。但在物理切片 $r=\widetilde{\beta}r_*$ 上，该场从谱中消失了，因为它的传播受到约束 (63) 的限制。因此，四维非局域系统中不存在这个鬼模，最终仅剩下一个传播自由度 $\phi(x)$ ，它只需要两个初始条件 $\phi(t_0,\mathbf{x})$ 和 $\dot{\phi}(t_0,\mathbf{x})$ ，外加局域系统在 $r_*=0$ 处对应解的信息 (见下文)。

Solutions

解

One might wonder why we have not set $\widetilde{\beta}=0$ in the first place. This choice plays a special role in finding actual solutions. In fact, to construct these solutions, one needs to specify a seed of the Φ field and then, using the diffusion equation, let this solution diffuse to the physical slice $r=\widetilde{\beta}r_*$. The most natural choice for this seed is the solution to the local system, i.e., setting $r_*=0$ in (54), such that

有人可能会好奇我们一开始为何不设定 $\widetilde{\beta}=0$ 。这个选择在寻找实际解的过程中起到特殊作用。事实上，构造这类解需要先给定 Φ 场的种子，随后利用扩散方程让这个解扩散到物理切片 $r=\widetilde{\beta}r_*$ 。该种子最自然的选择就是局域系统的解，即在 (54) 式中设定 $r_*=0$ ，由此可得

$$\Phi(0,t,\mathbf{x}) = \phi_{\text{local}}(t,\mathbf{x}). \tag{65}$$

Thus, the solution $\phi_{\text{sol}}(t,\mathbf{x})$ to the nonlocal system can be built out of $\phi_{\text{local}}(t,\mathbf{x})$ through a diffusion via the expression

因此，非局域系统的解 $\phi_{\text{sol}}(t,\mathbf{x})$ 可以通过扩散过程，由 $\phi_{\text{local}}(t,\mathbf{x})$ 按下式构造得到

$$\phi_{\text{sol}}(x) = \Phi\left(\tilde{\beta}r_*, x\right) = e^{\tilde{\beta}r_*\Box}\Phi(0,x) = \int_{-\infty}^{+\infty}\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}e^{-\tilde{\beta}r_*k^2}\tilde{\phi}_{\text{local}}(k),$$

(66)

where $\tilde{\phi}_{\text{local}}(k)$ is the Fourier transform of the solution ϕ_{local} of the local system, i.e.,

其中 $\tilde{\phi}_{\text{local}}(k)$ 是局域系统解 ϕ_{local} 的傅里叶变换，即

$$\Box\phi - V'(\phi) = 0, \tag{67}$$

so that plugging the formal solution (66) into the equation of motion (55) one finds the value for β.

因此将形式解 (66) 代入运动方程 (55)，就可以得到 β 的值。

To conclude, we cite two main reasons for the choice of ϕ_{local} as the seed of the diffusion.

最后，我们列出两点主要原因，说明为何选择 ϕ_{local} 作为扩散种子。

1. One expects to recover the solution of the local system when taking the limit $r_* \to 0$, and that is exactly what (66) guarantees.

1. 我们预期在取极限 $r_* \to 0$ 时可以回到局域系统的解，而 (66) 式正好保证了这一点。

2. Typically, diffusion does not alter the asymptotic behaviour of the field,

2. 通常情况下，扩散不会改变场的渐近行为，

$$\lim_{x\to\pm\infty}\phi_{\text{local}}(x) = \lim_{x\to\pm\infty}\phi_{\text{sol}}(x) \tag{68}$$

so that we may take the often known ϕ_{local} as a guide to construct the nonlocal solutions ϕ_{sol}.

因此我们可以用通常已知的 ϕ_{local} 作为参考，构造非局域解 ϕ_{sol}。

Solving a Paradox

悖论求解

At this point, one may be puzzled about the fate of the infinitely many initial conditions (50) expected in the nonlocal model. Technically, it is clear that they are given by

至此，读者可能会对非局域模型中预期存在的无穷多初始条件 (50) 的问题感到困惑。从形式上看，这些初始条件可写为

$$\phi^{(n)}(t_0, \mathbf{x}) = \Phi^{(n)}\left(\tilde{\beta} r_*, t_0, \mathbf{x}\right) \ \forall n \in \mathbb{N}, \tag{69}$$

where (n) is the n -th time derivative and the right-hand side is known after $\tilde{\beta}$ is determined by an algebraic equation. We started with infinitely many initial conditions that we encoded as one initial condition for the localized diffusing field Φ . However, from the point of view of the nonlocal four-dimensional system one still needs infinitely many conditions and, for different initial conditions, there should be different available solutions. How is it possible that this infinite number of initial values have been reduced to two?

其中 (n) 是 n 阶时间导数，当 $\tilde{\beta}$ 通过代数方程确定后，右侧即为已知。我们原本拥有无穷多初始条件，最终将其编码为局域扩散场 Φ 的单个初始条件。然而，从四维非局域系统的角度看，我们仍然需要无穷多条件，且不同初始条件应当对应不同的可行解。为什么无穷多初值会被约化为两个?

The problem with these questions is that they rely on the false premise that one could solve the four-dimensional Cauchy problem if one knew infinitely many initial conditions. But this is not possible because it leads to the paradox we mentioned above: If one knows all the initial conditions, one already knows the solution. The diffusion method explains the paradox. From the solution $\phi_{\text{local}}(x) = \Phi(0, x)$ of the local $(r_* = 0)$ four-dimensional system, via equation (66) one does know the solution $\Phi\left(\tilde{\beta} r_*, x\right)$ (except the value of $\tilde{\beta}$, easily found), so that one can compute all the initial conditions from (69). This knowledge, which looks to be needed 'in advance' and therefore unattainable from the nonlocal four-dimensional perspective, is a simple consequence of the diffusion equation from the localized five-dimensional perspective. In other words, the Cauchy problem of the nonlocal system is defined by the Cauchy problem of the localized diffusing system: this is the essence of the diffusion method. Without this definition, one is stuck with the impossibility of knowing a priori infinitely many initial conditions and, to the best of our knowledge, no alternative way out has been devised in the presence of interactions (In contrast, it has been known since the early days that, for linear nonlocal equations of motion, one can apply nonlocal field redefinitions and reduce the dynamics to a local one [37, 38] .).

这些问题的问题在于，它们建立在“只要知道无穷多初始条件就能求解四维柯西问题”这一错误前提上。但这并不成立，它恰恰会导出我们之前提到的悖论: 如果预先知道所有初始条件，那其实已经知道解了。扩散方法解释了这个悖论。从局域 $(r_* = 0)$ 四维系统的解 $\phi_{\text{local}}(x) = \Phi(0, x)$ 出发，通过方程 (66) 即可得到解 $\Phi\left(\tilde{\beta} r_*, x\right)$ (除 $\tilde{\beta}$ 的值外，其余都很容易求得)，因此我们可以通过 (69) 计算出所有初始条件。这种从四维非局域视角看需要“提前获知”因而无法得到的知识，从局域五维视角看只是扩散方程的简单推论。换句话说，非局域系统的柯西问题由局域扩散系统的柯西问题定义: 这就是扩散方法的核心。若没有这个定义，我们就会困在“不可能先验获知无穷多初始条件”的问题中，而且据我们所知，在存在相互作用的情况下目前还没有其他解决方法 (与之不同，人们早在很早之前就知道，对于线性非局域运动方程，可以通过非局域场重新定义将动力学约化为局域动力学 [37, 38] 。)。

As a cautionary note, the diffusion method vastly constrains all possible solutions to the nonlocal system but, due to the lack of injectivity of the map between the four-dimensional nonlocal system and the localized one with the extra direction, it also does not exclude the existence of other physical solutions to the nonlocal system not obtainable using this procedure. In other words, we do not know whether the diffusion method covers the whole space of admissible solutions. To date, there are no counter-examples indicating such a possibility.

需要提醒的是，扩散方法对非局域系统的所有可能解给出了极强的约束，但由于四维非局域系统和带额外维度的局域系统之间的映射不是单射，它也不能排除非局域系统存在其他无法通过该方法得到的物理解。换句话说，我们目前并不确定扩散方法是否覆盖了所有可行解的空间。迄今为止，还没有反例表明存在这类解。

Nonlocal Classical Gravity

非局域经典引力

Having shown how healthy nonlocal operators may affect the UV behaviour of the propagator as well as the way the diffusion method allows one to make sense of the initial conditions and the physical modes of the theory, we generalize this approach to gravity and derive the equations of motion of this theory, as well as some immediate consequences for Ricci-flat spacetimes. We also comment on the way nonlocal gravity addresses the singularity problem when dealing with black holes.

在阐明了良性非局域算子会如何影响传播子的紫外行为，以及扩散法使理论的初始条件和物理模变得有意义的方式之后，我们将这一方法推广到引力，推导得到该理论的运动方程，以及它对里奇平坦时空的一些直接推论。我们还探讨了非局域引力在处理黑洞问题时解决奇点问题的思路。

Action and Equations of Motion

作用量与运动方程

NLQG aims at solving the obstacles that Stelle's gravity (5) experiences in order to be considered a complete theory of quantum gravity. As we have discussed previously, the presence of higher-derivative terms turns out to be a problem at the classical and quantum level because of the violation of unitarity. These higher-derivative terms are quadratic in the curvature operators $\mathcal{R} = R_{\mu\nu\rho\sigma}, R_{\mu\nu}, R$ and their dependence on derivatives of the metric is

非局部经典引力旨在解决施泰勒引力 (5) 要成为完备量子引力理论所面临的障碍。正如我们此前讨论的，高导数项因违反幺正性，成为经典和量子层面都存在的问题。这些高导数项是曲率算符 $\mathcal{R} = R_{\mu\nu\rho\sigma}, R_{\mu\nu}, R$ 的二次项，它们对度规导数的依赖关系为

$$\mathcal{R} \sim \partial^2 g_{\mu\nu}, \partial g_{\mu\nu}^2 \Rightarrow \mathcal{R}^2 \sim \left(\partial^2 g_{\mu\nu}\right)^2, \left(\partial g_{\mu\nu}\right)^4, \partial^2 g_{\mu\nu} \partial g_{\mu\nu}^2.$$

As we will show in section "Nonlocal Quantum Gravity" at the tree level, the use of nonlocal operators in the action preserves unitarity, leading to a ghost-free theory. However, before we jump into quantum grounds, we formulate the classical theory of nonlocal gravity, whose action is given by

正如我们将在“非局部量子引力”一节中展示的，在树图阶，作用量中使用非局部算符可以保持幺正性，得到一个无鬼场的理论。但在进入量子讨论之前，我们先构建非局部引力的经典理论，其作用量由下式给出

$$S_{\mathrm{NLG}}=\frac{1}{2\kappa^{2}}\int d^{4}x\sqrt{-g}\left[R+R\gamma_{0}(\Box)R+R_{\mu\nu}\gamma_{2}(\Box)R^{\mu\nu}+R_{\mu\nu\rho\sigma}\gamma_{4}(\Box)R^{\mu\nu\rho\sigma}\right],\tag{70}$$

where $\gamma_0(\Box),\gamma_2(\Box)$, and $\gamma_4(\Box)$ are the form factors of this theory.

其中 $\gamma_0(\Box),\gamma_2(\Box)$ 和 $\gamma_4(\Box)$ 是该理论的形状因子。

For the purposes of this section, it is enough to choose a particular set of form factors,

就本节的目的而言，选择一组特定的形状因子即可，

$$\gamma_{2}(\Box)=-2\gamma_{0}(\Box)\equiv\gamma(\Box)\quad\gamma_{4}(\Box)=0.\tag{71}$$

With this choice, (70) becomes

基于这一选择，式 (70) 变为

$$S_{\mathrm{NLG}}=\frac{1}{2\kappa^{2}}\int d^{4}x\sqrt{-g}\left[R+G_{\mu\nu}\gamma(\Box)R^{\mu\nu}\right].\tag{72}$$

To derive the equations of motion of the action (72), one may either vary directly the action or consider an auxiliary field that does not introduce additional degrees of freedom on-shell and coincides with the original action. Let us recall both approaches.

要推导作用量 (72) 的运动方程，既可以直接对作用量变分，也可以引入一个辅助场——它不会在在壳时引入额外自由度，且与原作用量等价。我们下面回顾这两种方法。

Direct Computation

直接计算

By varying the action (72) in the presence of matter, i.e., $S=S_{\mathrm{NLG}}+S_{\mathrm{m}}$, one finds the equations of motion [35] (see also [39,40])

通过对存在物质时的作用量 (72) 做变分，即 $S=S_{\mathrm{NLG}}+S_{\mathrm{m}}$，可得运动方程 [35](另见 [39,40])

$$\kappa^{2}T_{\mu\nu}=e^{-r_{*}\Box}G_{\mu\nu}-\frac{1}{2}g_{\mu\nu}G_{\rho\sigma}\gamma(\Box)R^{\rho\sigma}+2G^{\rho}_{(\mu}\gamma(\Box)G_{\nu)\rho}+g_{\mu\nu}\nabla^{\rho}\nabla^{\sigma}\gamma(\Box)G_{\rho\sigma}$$

$$-2\nabla^{\rho}\nabla_{(\mu}\gamma(\Box)G_{\nu)\rho}+\frac{1}{2}\left(G_{\mu\nu}\gamma(\Box)R+R\gamma(\Box)G_{\mu\nu}\right)+\Theta_{\mu\nu}\left(R_{\rho\sigma},G^{\rho\sigma}\right),$$

(73)
where we introduced the energy-momentum tensor associated to the matter fields

此处我们引入了与物质场相关的能量动量张量

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}}, \tag{74}$$

and for Wataghin form factor

以及针对沃塔金形状因子

$$\Theta_{\mu\nu}\left(R_{\rho\sigma}, G^{\rho\sigma}\right) = -\int_0^{r_*} dq \widetilde{\Theta}_{\mu\nu}\left[e^{-q\Box}R_{\rho\sigma}, \frac{e^{-(r_*-q)\Box}-1}{\Box}G^{\rho\sigma}\right]. \tag{75}$$

The tensor $\widetilde{\Theta}_{\mu\nu}$ can be split into two parts, one symmetric and one anti-symmetric with respect to the arguments A and B, $\widetilde{\Theta}_{\mu\upsilon} = \widetilde{\Theta}_{\mu\upsilon}^{\text{sym}} + \widetilde{\Theta}_{\mu\upsilon}^{\text{antisym}}$, given by

张量 $\widetilde{\Theta}_{\mu\nu}$ 可拆分为两部分: 一部分关于自变量 A 和 B, $\widetilde{\Theta}_{\mu\upsilon} = \widetilde{\Theta}_{\mu\upsilon}^{\text{sym}} + \widetilde{\Theta}_{\mu\upsilon}^{\text{antisym}}$ 对称，另一部分反对称，形式如下

$$\widetilde{\Theta}_{\mu\nu}^{\text{sym}}\left(A_{\rho\sigma}, B^{\rho\sigma}\right) = -\nabla_\mu A_{\rho\sigma}\nabla_\nu B^{\rho\sigma} + \frac{1}{4}g_{\mu\nu}\nabla_\tau\left(A_{\rho\sigma}\nabla^\tau B^{\rho\sigma} + B^{\rho\sigma}\nabla_\tau A_{\rho\sigma}\right),$$

(76)

$$\begin{aligned}\widetilde{\Theta}_{\mu\upsilon}^{\text{antisym}}\left(A_{\rho\sigma}, B^{\rho\sigma}\right) = &\frac{1}{4}g_{\mu\upsilon}\nabla_\tau\left(A_{\rho\sigma}\nabla^\tau B^{\rho\sigma} - B^{\rho\sigma}\nabla^\tau A_{\rho\sigma}\right)\\ &+\nabla_\rho\left(A_{\mu\sigma}\nabla^\rho B_\upsilon^\sigma - B_\upsilon^\sigma\nabla^\rho A_{\mu\sigma}\right)\\ &+\nabla_\sigma\left(B_{\mu\rho}\nabla_\nu A^{\rho\sigma} - A^{\rho\sigma}\nabla_\nu B_{\mu\rho}\right)\\ &+\nabla_\rho\left(A_{\mu\sigma}\nabla_\upsilon B^{\sigma\rho} - B^{\sigma\rho}\nabla_\upsilon A_{\mu\sigma}\right).\end{aligned} \tag{77}$$

Here indices μ and υ are implicitly symmetrized. Notice that we recover Einstein’s equations if we set $\gamma(\Box) = 0$.

此处指标 μ 和 υ 默认做对称化处理。不难发现，若令 $\gamma(\Box) = 0$ ，我们将回到爱因斯坦方程。

Auxiliary Field

辅助场

Alternatively, we can derive the equations of motion introducing an auxiliary rank-2 symmetric tensor $\phi_{\mu\nu}$ in the action [35]:

我们也可以在作用量中引入一个辅助二阶对称张量 $\phi_{\mu\nu}$ 来推导运动方程 [35]:

$$\tilde{S}\left[g,\phi\right]=\frac{1}{2\kappa^{2}}\int d^{4}x\sqrt{-g}\left[R+\left(2R_{\mu\nu}-\phi_{\mu\nu}+\frac{1}{2}g_{\mu\nu}\phi\right)\gamma\left(\Box\right)\phi^{\mu\nu}\right], \tag{78}$$

where $\phi=\phi_{\alpha}^{\alpha}$, and the respective equations of motion for both fields:

其中为 $\phi=\phi_{\alpha}^{\alpha}$ ，两个场各自的运动方程为:

$$\frac{\delta\tilde{S}}{\delta g_{\mu\nu}}=0,\ \frac{\delta\tilde{S}}{\delta\phi_{\mu\nu}}=0, \tag{79}$$

are [35]

参见文献 [35]

$$\begin{aligned}\kappa^{2}T_{\mu\nu}=G_{\mu\nu}+\Box\gamma\left(\Box\right)\phi_{\mu\nu}-\frac{1}{2}g_{\mu\nu}X_{\rho\sigma}\gamma\left(\Box\right)\phi^{\rho\sigma}+2\phi_{(\mu}^{\rho}\gamma\left(\Box\right)\phi_{\upsilon)\rho}\\+g_{\mu\nu}\nabla^{\rho}\nabla^{\sigma}\gamma\left(\Box\right)\phi_{\rho\sigma}-2\nabla^{\rho}\nabla_{(\mu}\gamma\left(\Box\right)\phi_{\nu)\rho}\\-\frac{1}{2}\left(\phi_{\mu\upsilon}\gamma\left(\Box\right)\phi+\phi\gamma\left(\Box\right)\phi_{\mu\upsilon}\right)+\Theta_{\mu\upsilon}\left(X_{\rho\sigma},\phi^{\rho\sigma}\right),\end{aligned} \tag{80}$$

$$\phi_{\mu\upsilon}=G_{\mu\upsilon} \tag{81}$$

where we have defined

其中我们定义了

$$X_{\mu\upsilon}=2R_{\mu\upsilon}-\phi_{\mu\upsilon}+\frac{1}{2}g_{\mu\upsilon}\phi. \tag{82}$$

Note that we can calculate the trace of (81) on-shell,

注意，我们可以在壳计算式 (81) 的迹,

$$\phi=G=-R,\ X_{\mu\upsilon}=R_{\mu\upsilon}, \tag{83}$$

and we recover (73).

即可得到式 (73)。

Diffusion Method for Nonlocal Gravity

非局域引力的扩散法

Similarly to what we did in section ”Diffusion Method”, one can show that this nonlocal formulation of gravity does not bring ghost modes to the particle spectrum of the theory and the initial conditions problem is solved consistently. We take the minimal operator (34), referring the reader to [36] for the case of asymptotically polynomial form factors.

与我们在“扩散法”一节中所做的工作类似，我们可以证明: 引力的这种非局域表述不会给理论的粒子谱引入鬼模，且初值问题可以得到一致求解。本文采用最小算符 (34)，渐近多项式形状因子的情况读者可参考文献 [36]。

The action that one must take into account is given by [35]

需要考虑的作用量由文献 [35] 给出

$$S\left[\Phi, g, \chi, \lambda\right] = \frac{1}{2\kappa^2}\int d^4x dr\sqrt{-g}\left(\mathcal{L}_R + \mathcal{L}_\Phi + \mathcal{L}_\chi + \mathcal{L}_\lambda\right), \tag{84}$$

with

其中

$$\mathcal{L}_R = R\left(r\right) \tag{85}$$

$$\mathcal{L}_\Phi = -\int_0^{r_*} ds\left[2\widetilde{\mathcal{R}}_{\mu\nu}\left(r\right) - \Phi\left(r\right) + \frac{1}{2}g_{\mu\nu}\left(r\right)\Phi\left(r\right)\right]\Phi^{\mu\nu}\left(r-s\right), \tag{86}$$

$$\mathcal{L}_\chi = -\int_0^{r_*} ds\int_0^{s}\chi_{\mu\nu}\left(r-q\right)\left(\partial_{r'} - \Box\right)\Phi^{\mu\nu}\left(r'\right), \tag{87}$$

$$\mathcal{L}_\lambda = \lambda_{\mu\nu}\partial_r g^{\mu\nu}\left(r\right), \tag{88}$$

where we have omitted the x -dependence in all fields, and $\widetilde{\mathcal{R}}_{\mu\nu}$ is the Ricci tensor associated to $g_{\mu\nu}\left(r\right)$, that on-shell will become r -independent. In fact

我们在所有场中省略了对 x 的依赖，$\widetilde{\mathcal{R}}_{\mu\nu}$ 是与 $g_{\mu\nu}\left(r\right)$ 对应的里奇张量，它在在壳条件下将变为与 r 无关。事实上

$$\frac{\delta S}{\delta\lambda_{\mu\nu}} = 0 \Rightarrow \partial_r g^{\mu\nu} = 0 \Rightarrow g_{\mu\nu}\left(r, x\right) = g_{\mu\nu}\left(x\right). \tag{89}$$

From this localized Lagrangian, we can obtain the equations of motion:

由这个定域拉格朗日量，我们可以得到运动方程:

$$(\partial_r - \Box)\,\Phi_{\mu\nu} = 0, \tag{90}$$

$$(\partial_r - \Box)\,\chi_{\mu\nu} = 0, \tag{91}$$

$$\int_0^{r_*} ds \left[X_{\mu\upsilon}(r-s) + X_{\mu\upsilon}(r+s) - 2\widetilde{\mathcal{R}}_{\mu\upsilon}(r-s) + \chi_{\mu\upsilon}(r-s) - \chi_{\mu\upsilon}(r+s)\right] = 0,$$

(92)

$$\kappa^2 T_{\mu\upsilon} = G_{\mu\upsilon} - \int_0^{r_*} ds \Big\{ -\frac{1}{2} g_{\mu\upsilon} X_{\alpha\beta}(r)\,\Phi^{\alpha\beta}(r-s) + 2\Phi_{\sigma(\mu}(r)\,\Phi^{\sigma}_{\upsilon)}(r-s)$$

$$+\Box\Phi_{\mu\upsilon}(r-s) + g_{\mu\upsilon}\nabla^{\sigma}\nabla^{\tau}\Phi_{\sigma\tau}(r-s) - e\nabla^{\sigma}\nabla_{(\mu}\Phi\upsilon)\sigma(r-s)$$

$$-\frac{1}{2}\left[\Phi_{\mu\upsilon}(r)\,\Phi(r-s) + \Phi(r)\,\Phi_{\mu\upsilon}(r-s)\right]$$

$$-\int_0^{s} dq\overline{\Theta}_{\mu\upsilon}\left[\chi_{\sigma\tau}(r-q), \Phi^{\sigma\tau}(r+q-s)\right]\Big\}$$

(93)

where $X_{\mu\nu}$ is

其中 $X_{\mu\nu}$ 为

$$X_{\mu\upsilon}(r) = 2\widetilde{\mathcal{R}}_{\mu\upsilon}(r) - \Phi_{\mu\upsilon}(r) + \frac{1}{2} g_{\mu\upsilon}(r)\,\Phi(r). \tag{94}$$

From the first two equations, we see that the auxiliary fields $\Phi_{\mu\nu}$ and $\chi_{\mu\nu}$ propagate via a diffusion equation. Similarly to what we did in the scalar field theory, we may impose by hand a constraint at the slice $r = \widetilde{\beta} r_*$ analogous to (63),

从前两个方程我们可以看出，辅助场 $\Phi_{\mu\nu}$ 和 $\chi_{\mu\nu}$ 通过扩散方程传播。与我们在标量场论中所做的类似，我们可以在切片 $r = \widetilde{\beta} r_*$ 上人为施加一个类 (63) 式的约束，

$$\chi_{\mu\upsilon}\left(\widetilde{\beta} r_*\right) = X_{\mu\upsilon}\left(\widetilde{\beta} r_*\right) = R_{\mu\upsilon} \;\Rightarrow\; \Phi_{\mu\upsilon}\left(\widetilde{\beta} r_*\right) = \phi_{\mu\upsilon} = G_{\mu\upsilon}, \tag{95}$$

which satisfies (92). Using the diffusion equation for $\Phi_{\mu\nu}$, we have

该约束满足式 (92)。利用 $\Phi_{\mu\nu}$ 的扩散方程，我们得到

$$-\int_0^{r_*} ds\Phi_{\mu\upsilon}(r-s) = \frac{e^{-r_*\Box} - 1}{\Box}\Phi_{\mu\upsilon}(r) = \gamma(\Box)\,\Phi_{\mu\upsilon}(r), \tag{96}$$

so that at the physical slice $\widetilde{\beta} r_*$ we recover the equations of motion (73) of the nonlocal theory.

因此在物理切片 $\widetilde{\beta} r_*$ 上，我们可以重新得到非局域理论的运动方程 (73)。

The equivalence between the localized five-dimensional system at the slice $r = \widetilde{\beta} r_*$ and the nonlocal four-dimensional theory with the minimal form factor can be generalized to other exponential-monomial form factors such as Krasnikov's. However, when dealing with asymptotically polynomial form factors one has to follow a different approach [36] where a diffusion-like equation is implemented in a more sophisticated way.

切片 $r = \widetilde{\beta} r_*$ 处定域化五维系统与带有最小形状因子的非局域四维理论之间的等价性，可以推广到其他指数-单项式形状因子，比如克拉斯尼科夫形状因子。不过，处理渐近多项式形状因子时需要采用不同的方法 [36]，该方法中类扩散方程的实现更为复杂。

Initial Conditions, Degrees of Freedom and Absence of Ghosts

初值条件、自由度与无鬼场

In the gravitational case, we have to introduce three additional fields: $\Phi_{\mu\nu}, \chi_{\mu\nu}$, and $\lambda_{\mu\nu}$. The latter was simply used to implement the condition $\partial_r g_{\mu\nu}$ and does not show any dynamics by itself. On the other hand, the field $\chi_{\mu\nu}$ has a similar origin that the scalar field χ introduced in (58). In a similar vein, we have constrained by hand this $\chi_{\mu\nu}$ by the expression (95), so that $\chi_{\mu\nu} = R_{\mu\nu} \sim \partial g^2_{\mu\nu}, \partial^2 g_{\mu\nu}$ as well as $\Phi_{\mu\nu}$ on-shell by (81), so that they do not become additional propagating modes, since their diffusion is frozen in the $\widetilde{\beta} r_*$ slice.

在引力情形下，我们需要引入三个额外场: $\Phi_{\mu\nu}, \chi_{\mu\nu}$ 和 $\lambda_{\mu\nu}$。后者仅用于实现条件 $\partial_r g_{\mu\nu}$，本身不存在任何动力学。而场 $\chi_{\mu\nu}$ 的起源与 (58) 式中引入的标量场 χ 类似。同理，我们通过表达式 (95) 人工约束了该 $\chi_{\mu\nu}$，使得 $\chi_{\mu\nu} = R_{\mu\nu} \sim \partial g^2_{\mu\nu}, \partial^2 g_{\mu\nu}$ 和 $\Phi_{\mu\nu}$ 在壳满足 (81) 式，因此它们不会成为额外传播模式，因为它们的扩散在 $\widetilde{\beta} r_*$ 切片中被冻结。

From this relation between $\chi_{\mu\nu}$ and the Ricci tensor, we conclude that for the minimal/Wataghin form factor we only need four initial conditions

根据 $\chi_{\mu\nu}$ 与里奇张量之间的关系，我们可以得出结论: 对于极小/沃特金形状因子，我们仅需要四个初值条件

$$g_{\mu\nu}(t_0, \mathbf{x}),\ \dot{g}_{\mu\nu}(t_0, \mathbf{x}),\ \ddot{g}_{\mu\nu}(t_0, \mathbf{x}),\ \dddot{g}_{\mu\nu}(t_0, \mathbf{x}), \tag{97}$$

instead of an infinite number of them. This number of initial values can increase for other types of form factors but it remains finite.

而非无穷多个。对于其他类型的形状因子，初值的数量可能增加，但始终是有限的。

We also have to count the physical degrees of freedom of this theory, and to do so we have to find the propagating independent components of the tensorial fields of our equations. First of all, let us recall how many physical degrees of freedom contains the graviton: since it is a rank-2 symmetric tensor, in four dimensions it has 10 independent components, but gauge invariance coming from diffeomorphism invariance reduces them to 6 . Finally, the contracted Bianchi identities $\nabla_\mu G^{\mu\nu}=0$ reduce by 4 this amount, giving rise to only 2 independent components of the graviton.

我们还需要计算该理论的物理自由度，为此我们需要找出我们方程中张量场的独立传播分量。首先，我们回顾引力子包含多少物理自由度：引力子是二阶对称张量，在四维空间中它有 10 个独立分量，但微分同胚不变性带来的规范不变性将其缩减为 6 个。最后，缩并比安基恒等式 $\nabla_\mu G^{\mu\nu}=0$ 再次将该数量减少 4，最终引力子仅剩下 2 个独立分量。

On the other hand, for the auxiliary field $\phi_{\mu\nu}=G_{\mu\nu}$ on-shell, only the Bianchi identities apply, so that it contains 6 degrees of freedom. In conclusion, we have in total $6+2=8$ propagating degrees of freedom, result that coincides with the counting in Stelle's gravity [10]. The main difference here is that nonlocal gravity does not contain any ghost field so that the theory is stable. In fact, one may follow the perturbative procedure described in [10] to show that, expanding the Lagrangian at second order in the perturbation/graviton field $h_{\mu\nu}$ defined as

另一方面，对于在壳的辅助场 $\phi_{\mu\nu}=G_{\mu\nu}$，仅需满足比安基恒等式，因此它包含 6 个自由度。综上，我们总共有 $6+2=8$ 个传播自由度，该结果与施泰勒引力中的计数一致 [10]。此处最主要的区别是非局域引力不包含任何鬼场，因此该理论是稳定的。事实上，我们可以遵循 [10] 中描述的微扰方法证明，将拉格朗日量按微扰/引力子场 $h_{\mu\nu}$ 做二阶展开，定义如下

$$g_{\mu\upsilon}=\eta_{\mu\upsilon}+2\kappa h_{\mu\upsilon}, \tag{98}$$

the ghost disappears from the particle spectrum of the nonlocal theory. This Lagrangian at second order has the following expression for a generic form factor (33) [36]:

鬼场会从非局域理论的粒子谱中消失。对于一般形状因子 (33)，该二阶拉格朗日量的表达式如下 [36]:

$$\mathcal{L}^{(2)}=\mathcal{L}_E\left(h_{\mu\upsilon}\right)+3\phi\frac{\Box}{1-e^{-\mathrm{H}(\Box)}}\phi-\frac{1}{2}\psi_{\mu\upsilon}\left(\eta^{\mu\rho}\eta^{\upsilon\sigma}-\eta^{\mu\upsilon}\eta^{\rho\sigma}\right)\frac{\Box}{1-e^{-\mathrm{H}(\Box)}}\psi_{\rho\sigma},$$

(99)

where $\psi_{\mu\upsilon}$ is the traceless part of $\phi_{\mu\upsilon}$, and ϕ is its trace, and $\mathcal{L}_E$ is the Einstein’s linearized Lagrangian given by [41]

其中 $\psi_{\mu\upsilon}$ 是 $\phi_{\mu\upsilon}$ 的无迹部分，ϕ 是它的迹，$\mathcal{L}_E$ 是爱因斯坦线性化拉格朗日量，由 [41] 给出

$$\mathcal{L}_E\left(h_{\mu\nu}\right)=\frac{1}{2}h_{\mu\nu}\Box h^{\mu\nu}-\frac{1}{2}h\Box h+h^{\mu\nu}\partial_\mu\partial_\nu h-h^{\mu\nu}\partial_\rho\partial_\nu h^\rho_\mu. \tag{100}$$

From these expressions, one sees that the second and the third term in (99) give rise to the mass of the additional modes ϕ and $\psi_{\mu\nu}$. Stelle’s gravity can be interpreted as a truncated expansion of the minimal/Wataghin form factor, in particular, one recovers Stelle's particle spectrum in the limit

从这些表达式可以看出，(99) 式中的第二项和第三项给出了额外模式 ϕ 和 $\psi_{\mu\nu}$ 的质量。施泰勒引力可以解释为极小/沃特金形状因子的截断展开，特别地，在该极限下我们可以重新得到施泰勒的粒子谱

$$\frac{e^{\mathrm{H}(\Box)}}{\gamma(\Box)} = \frac{\Box}{1-e^{-\mathrm{H}(\Box)}} \simeq \frac{\Box}{\mathrm{H}(\Box)} \simeq 1, \tag{101}$$

giving rise to propagators with the wrong sign, i.e., ghost fields.

最终得到符号错误的传播子，即鬼场。

Nevertheless, the nonlocal operator (101) cannot be truncated in our gravitational theory. In particular, the propagator of the fields $\psi_{\mu\nu}$ and ϕ will be proportional to $\gamma e^{-\mathrm{H}}$ and, by definition of the form factors, it will never display new extra poles in the theory.

然而，我们的引力理论中不能对非局域算符 (101) 做截断。特别地，场 $\psi_{\mu\nu}$ 和 ϕ 的传播子与 $\gamma e^{-\mathrm{H}}$ 成正比，根据形状因子的定义，它永远不会在理论中产生新的额外极点。

Solutions

解

In analogy with what we did in the scalar field case, we can construct exact or approximate solutions of the nonlocal system using as a seed of the diffusion equation the local system, i.e., taking $\gamma(\Box) = 0 \Rightarrow r_* = 0$, such that the equations of motion are given by the Einstein's equations

类比我们在标量场情形下的操作，我们可以将局域系统作为扩散方程的种子，构造非局部系统的精确解或近似解，也就是取 $\gamma(\Box) = 0 \Rightarrow r_* = 0$ ，使得运动方程由爱因斯坦方程给出

$$\kappa^2 T_{\mu\nu} = G_{\mu\nu} \Rightarrow \chi_{\mu\nu}(0,x) = R^{\mathrm{local}}_{\mu\nu}(x), \ \Phi_{\mu\nu}(0,x) = G^{\mathrm{local}}_{\mu\nu}(x),$$

(102)

where $R^{\mathrm{local}}_{\mu\nu}$ and $G^{\mathrm{local}}_{\mu\nu}$ are built with a solution $g^{\mathrm{local}}_{\mu\nu}$ of the local system. From these local solutions, one can diffuse to the physical slice $r \to \tilde{\beta} r_*$. However, one must note that in a curved spacetime, the integral representation of the kernel (22) is no longer correct since the functions $\exp(\pm ik \cdot x)$ are not eigenfunctions of the Laplace-Beltrami operator. In this case, one has to find two eigenfunctions of $\Box$ and write (22) as their linear superposition [31], as explained in section "Representations of Form Factors".

其中 $R^{\mathrm{local}}_{\mu\nu}$ 和 $G^{\mathrm{local}}_{\mu\nu}$ 由定域系统的解 $g^{\mathrm{local}}_{\mu\nu}$ 构造得到。利用这些定域解，我们可以扩散到物理切片 $r \to \tilde{\beta} r_*$ 。但需要注意，在弯曲时空中，核的积分表示式 (22) 不再成立，因为函数 $\exp(\pm ik \cdot x)$ 不是拉普拉斯-贝尔特拉米算子的本征函数。在这种情况下，我们需要找到 $\Box$ 的两个本征函数，并将 (22) 写为它们的线性叠加 [31]，这一点我们已经在“形状因子的表示”一节中说明。

Analysis and Properties of NLG

非局域引力 (NLG) 的分析与性质

Once derived the equations of motion of the minimal nonlocal gravitational theory (72) we may explore some of its classical properties. In general, one expects that finding analytic solutions to these equations will not be possible. However, the structure of the equations (73) allows us to say a few things about its solutions when considering Ricci-flat spacetimes. We can also prove the stability of these solutions under small perturbations as we have recently done in (98) on Minkowski but including more general backgrounds where the additional degrees of freedom $\psi_{\mu\nu}$ and ϕ can propagate. Lastly, we check that the stability of these backgrounds can be generalized to any perturbative order.

得到最小非局域引力理论的运动方程 (72) 后，我们可以探究它的若干经典性质。一般而言，这类方程很难找到解析解。不过，方程 (73) 的结构允许我们在讨论里奇平坦时空时，对其解得出一些结论。我们还可以证明这些解在小扰动下稳定，正如我们此前在文献 (98) 中对闵氏空间所做的，我们还纳入了附加自由度 $\psi_{\mu\nu}$ 和 ϕ 可传播的更一般背景。最后，我们验证了这些背景的稳定性可以推广到任意微扰阶。

Ricci-Flat Spacetimes

里奇平坦时空

The first question we may ask ourselves is whether it is possible to export the solutions of Einstein's gravity to NLG. The very structure of the equations of motion (73) tells us that it is possible, in particular, if we consider Ricci-flat spacetimes or vacuum solutions of Einstein's gravity,

我们首先会思考，爱因斯坦引力的解能否推广到非局部引力 (NLG) 中。运动方程 (73) 的结构本身告诉我们这是可行的，尤其是当我们考虑爱因斯坦引力中的里奇平坦时空或真空解时，

$$R_{\mu\nu} = 0 \;\Rightarrow\; G_{\mu\nu} = 0 \;\Rightarrow\; T_{\mu\nu} = 0, \tag{103}$$

then the equations of motion of NLG are automatically satisfied so that they constitute valid solutions of the nonlocal theory, such as Minkowski, Schwarzschild or Kerr spacetime.

非局部引力的运动方程会自动成立，因此这些时空都是非局部引力理论的有效解，例如闵可夫斯基时空、史瓦西时空和克尔时空。

NLG aspires to avoid the singularities arising in GR, that tell us that the theory breaks down in some region of spacetime. This problem was initially identified by Schwarzschild [42] and Hilbert [43] when they studied the metric nowadays taking the former's name:

非局部引力 (NLG) 旨在消除广义相对论 (GR) 中出现的奇点——奇点意味着理论在部分时空区域失效。这个问题最早由史瓦西 [42] 和希尔伯特 [43] 在研究如今以史瓦西命名的度规时发现：

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

(104)

This metric blows up at $r = 2GM$ and $r = 0$, but one easily shows that, while the first singularity is caused by an inappropriate choice of coordinates, the second is an $O(1/r)$ curvature singularity that cannot be removed by a suitable coordinate transformation. Furthermore, in the non-relativistic limit, the Schwarzschild metric reproduces Newtonian gravity after linearizing the metric

该度规在 $r = 2GM$ 和 $r = 0$ 处发散，但不难证明，第一个奇点是坐标选取不当导致的，第二个则是无法通过合适的坐标变换消除的 $O(1/r)$ 曲率奇点。此外，在非相对论极限下，对史瓦西度规做线性化后可以得到牛顿引力

$$ds^2 = -[1 + 2\Phi(\mathbf{x})]\,dt^2 + [1 - 2\Phi(\mathbf{x})]\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{105}$$

such that the Newtonian potential $\Phi(\mathbf{x})$ satisfies the Poisson’s equation

因此牛顿势 $\Phi(\mathbf{x})$ 满足泊松方程

$$\partial_i\partial^i\Phi = \delta^{(3)}(\mathbf{x}). \tag{106}$$

Stability

稳定性

We say that a background solution $g^{(0)}_{\mu\nu}$ is stable against linear perturbations $h_{\mu\nu}$ if the metric $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$ does not blow up when it solves the vacuum equations of motion, i.e., if the perturbation $h_{\mu\nu}$ remains small throughout the dynamical evolution. From this definition, one may prove that Minkowski [44] and Schwarzschild (104) [45] spacetimes are stable in GR.

我们称背景解 $g^{(0)}_{\mu\nu}$ 对线性微扰 $h_{\mu\nu}$ 是稳定的: 若度规 $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$ 在满足真空运动方程时不会发散，即微扰 $h_{\mu\nu}$ 在整个动力学演化过程中始终保持很小。根据该定义，可以证明闵氏时空 [44] 和史瓦西时空 (104)[45] 在广义相对论中是稳定的。

Furthermore, when $\gamma_4 = 0$ in the action (70), we also have that Schwarzschild [46] and Minkowski [47] spacetimes are stable against linear perturbations in NLQG, and that this result can be generalized to all Ricci-flat spacetimes [48] and all perturbative orders [49]. Note, however, that this argument is based on the results exposed in section ”Ricci-Flat Spacetimes” in which we see the direct link between Einstein’s gravity and NLG in Ricci-flat spacetimes, so that stability in GR is inherited by the nonlocal theory.

此外，当 $\gamma_4 = 0$ 作用于作用量 (70) 时，史瓦西时空 [46] 和闵氏时空 [47] 在非局部量子引力中对线性微扰也是稳定的，该结果还可以推广到所有里奇平坦时空 [48] 和所有微扰阶 [49]。但需要注意，该论证基于“里奇平坦时空”一节给出的结论: 在里奇平坦时空中，爱因斯坦引力与非局部引力存在直接关联，因此广义相对论中的稳定性可以被非局部理论继承。

To prove the latter statement, it is convenient to introduce the Lichnerowicz operator Δ_L , that acting on a rank-2 symmetric field $\psi_{\mu\nu}$ is defined as

为了证明上述结论，引入利赫诺维奇算子 Δ_L 会更方便，该算子作用在二阶对称场 $\psi_{\mu\upsilon}$ 上的定义为

$$\Delta_L\psi_{\mu\upsilon} = 2R^{\rho}_{\mu\upsilon\tau}\psi^{\tau}_{\rho} + R_{\mu\rho}\psi^{\rho}_{\upsilon} + R_{\sigma\upsilon}\psi^{\sigma}_{\mu} - \Box\psi_{\mu\upsilon}, \tag{107}$$

and acting on a scalar it is simply given by $\Delta_L\psi = -\Box\psi$.

作用在标量上时，它可以简单表示为 $\Delta_L\psi = -\Box\psi$ 。

Along with this new operator we can define a theory formally equivalent to (70) but with the substitution $-\Box \to \Delta_L$ everywhere, namely

配合这个新算子，我们可以定义一个形式上等价于 (70) 的理论，只需处处替换为 $-\Box \to \Delta_L$ ，即

$$\mathcal{L} = R + G_{\mu\nu}\gamma\left(\Delta_L\right)R^{\mu\nu}. \tag{108}$$

Clearly, this theory and the minimal theory (72) share the same renormalization properties on Minkowski spacetime and, for simplicity, we use the Lagrangian (108) to prove stability order by order. The proof of the stability for the minimal theory is more involved [47] but, since the perturbative expansion around Minkowski spacetime is the same for both theories, the result reviewed here is applicable to the theory of our interest.

显然，该理论与最小理论 (72) 在闵氏时空上具有相同的重整化性质，为简化起见，我们使用拉格朗日量 (108) 逐阶证明稳定性。最小理论的稳定性证明更为复杂 [47]，但由于两种理论在闵氏时空周围的微扰展开是相同的，本文梳理的结果同样适用于我们关注的理论。

The equations of motion of this nonlocal theory are given by [48]

该非局部理论的运动方程为 [48]

$$e^{H(\Delta_L)}G_{\mu\nu} + \mathcal{O}_{\mu\nu}\left(\mathcal{R}^2\right) = 0, \tag{109}$$

where $\mathcal{O}_{\mu\upsilon}\left(\mathcal{R}^2\right)$ denotes terms at least quadratic in the Ricci tensor $R_{\mu\upsilon}$. This form of the equations of motion is compatible with the solution $R_{\mu\upsilon} = 0$. One can directly substitute $R_{\mu\upsilon} = G_{\mu\upsilon}$ and acting $e^{-H(\Delta_L)}$ on both sides of (109), one finds the nested expression

其中 $\mathcal{O}_{\mu\upsilon}\left(\mathcal{R}^2\right)$ 表示至少是里奇张量 $R_{\mu\upsilon}$ 二次方的项。这种形式的运动方程与解 $R_{\mu\upsilon} = 0$ 相容。我们可以直接代入 $R_{\mu\upsilon} = G_{\mu\upsilon}$ ，再对 (109) 的两侧作用 $e^{-H(\Delta_L)}$ ，得到嵌套表达式

$$G_{\mu\upsilon} = e^{-H(\Delta_L)}\mathcal{O}_{\mu\upsilon}\left(\mathcal{G}^2\right), \tag{110}$$

where $\mathcal{O}_{\mu\upsilon}\left(\mathcal{G}^2\right)$ denotes terms at least quadratic in the Einstein tensor $G_{\mu\upsilon}$,

其中 $\mathcal{O}_{\mu\upsilon}\left(\mathcal{G}^2\right)$ 表示至少是爱因斯坦张量 $G_{\mu\upsilon}$ 二次方的项,

$$\mathcal{O}_{\mu\nu}\left(\mathcal{G}^2\right) = \left[\mathcal{G}\mathcal{O}\mathcal{G}\right]_{\mu\nu} + \mathcal{O}_{\mu\nu}\left(\mathcal{G}^3\right), \tag{111}$$

being $\mathcal{O}$ an operator that acts on $G_{\mu\nu}$ in both sides. Taking the perturbative expansion

其中 $\mathcal{O}$ 是一个作用在方程两侧 $G_{\mu\nu}$ 上的算子。对其进行微扰展开

$$g_{\mu\upsilon}=\sum_{n=0}^{\infty}\varepsilon^{n}h_{\mu\upsilon}^{(n)},\;h_{\mu\upsilon}^{(0)}=g_{\mu\upsilon}^{(0)}, \tag{112}$$

with $\varepsilon \ll 1$, the Einstein tensor $G_{\mu\nu}$ to all orders is

在 $\varepsilon \ll 1$ 的条件下，所有阶的爱因斯坦张量 $G_{\mu\nu}$ 可写为

$$G_{\mu\nu}\left(g_{\mu\nu}\right)=\sum_{n=0}^{\infty}\varepsilon^{n}G_{\mu\nu}^{(n)}, \tag{113}$$

where $G_{\mu\nu}^{(0)}=0$ because the background metric is Ricci-flat. Also, the exponential operator is abstractly written as

其中 $G_{\mu\nu}^{(0)}=0$ ，因为背景度规是里奇平坦的。另外，指数算子可以抽象记为

$$e^{-H(\Delta_L)}=\sum_{n=0}^{\infty}\varepsilon^{n}S^{(n)}. \tag{114}$$

Since $\mathcal{G}\left(g_{\mu\upsilon}\right)\sim\varepsilon$, we may neglect cubic terms in the expansion (111) and, eliminating the tensorial structure of the previous expressions, we can write (110) order by order as

由于 $\mathcal{G}\left(g_{\mu\upsilon}\right)\sim\varepsilon$ ，我们可以忽略展开式 (111) 中的三次项，消去前文表达式中的张量结构后，我们可以将 (110) 逐阶写为

$$\mathcal{G}^{(n)}=\sum_{h=0}^{n}\sum_{k=0}^{h}\sum_{q=0}^{k}S^{(n-h)}\mathcal{G}^{(h-k)}\mathcal{O}^{(k-q)}\mathcal{G}^{(q)}. \tag{115}$$

By recursion, one obtains that $\mathcal{G}^{(n)}=0\forall n\in\mathbb{N}$. For instance, taking $n=1$ one gets

通过递推可得 $\mathcal{G}^{(n)}=0\forall n\in\mathbb{N}$ 。例如，代入 $n=1$ 可得

$$\mathcal{G}^{(1)}=S^{(0)}\left(\mathcal{G}^{(1)}\mathcal{O}^{(0)}\mathcal{G}^{(0)}+\mathcal{G}^{(0)}\mathcal{O}^{(1)}\mathcal{G}^{(0)}+\mathcal{G}^{(0)}\mathcal{O}^{(0)}\mathcal{G}^{(1)}\right)+S^{(1)}\mathcal{G}^{(0)}\mathcal{O}^{(0)}\mathcal{G}^{(0)}=0.$$

(116)

Therefore, one concludes that the solutions of the nonlocal theory (108) are stable at any order if they are stable in Einstein's theory.

因此可以得到结论: 若解在爱因斯坦理论中是稳定的，那么它在非局部理论 (108) 中任意阶都是稳定的。

This result also holds for the theory (72) [47] and has immediate consequences on the discussion about the additional 6 degrees of freedom $\psi_{\mu\nu}$ and ϕ of section "Diffusion Method for Nonlocal Gravity". There, we showed that they did not propagate in Minkowski spacetime, but now we realize that this is true on any Ricci-flat metric. In particular, on these backgrounds the stability of the nonlocal theory is inherited from Einstein's theory, in which there are no additional propagating degrees of freedom other than the two corresponding to the graviton. Consequently, the ghost modes $\psi_{\mu\nu}$ and ϕ are not dynamical fields and the theory is unitary.

这一结果同样适用于理论 (72) [47]，并且对“非局域引力的扩散方法”一节中关于额外 6 个自由度 $\psi_{\mu\nu}$ 和 ϕ 的讨论有直接影响。我们此前已经证明，这两个自由度不在闵可夫斯基时空传播，而现在我们可以得出结论: 该结论在任意里奇平坦度规上都成立。具体而言，在这些背景下，非局域理论的稳定性继承自爱因斯坦理论——爱因斯坦理论中除引力子对应的两个自由度外，不存在额外的传播自由度。因此，鬼模 $\psi_{\mu\nu}$ 和 ϕ 不是动力学场，该理论是幺正的。

Smoothing Out Singularities

消除奇点

To wrap up this part on the classical theory, we recall a key property of weakly nonlocal operators that consists in smearing the singularities, which makes this theory so appealing in order to deal with the singularity problem arising in GR. After showing how nonlocality is able to smooth out classical divergences, we briefly discuss implications for black holes.

为了总结经典理论这一部分，我们回顾弱非局域算符的一个核心性质: 它可以弥散奇点，这也是该理论在解决广义相对论奇点问题上极具吸引力的原因。我们将说明非局域性如何平滑经典发散，之后简要讨论其对黑洞的启示。

Let us start by some preliminaries that illustrate the manner form factors avoid singularities. First of all, consider the Poisson equation for the static gravitational potential $\Phi(\mathbf{x})$ in three-dimensional flat space,

我们先从一些预备知识入手，说明形状因子规避奇点的方式。首先，考虑三维平坦空间中静态引力势 $\Phi(\mathbf{x})$ 的泊松方程,

$$\gamma(\Box)\,\Phi(\mathbf{x}) = \delta^{(3)}(\mathbf{x})\,, \tag{117}$$

where $\Box = \partial_i\partial^i$. There are three cases of interest: the classical second-derivative operator, higher-derivative (HD) operators and nonlocal form factors (Fig. 4).

其中为 $\Box = \partial_i\partial^i$ 。我们有三种值得关注的情形: 经典二阶导数算符、高阶导数 (HD) 算符和非局域形状因子 (图 4)。

- Standard Yukawa potential. In this case, $\gamma(\Box) = \Box - m^2$ and one can calculate $\Phi(\mathbf{x})$ using spherical coordinates:

- 标准汤川势。在该情形下，$\gamma(\Box) = \Box - m^2$ ，我们可以利用球坐标计算 $\Phi(\mathbf{x})$:

$$(\square - m^2)\,\Phi(\mathbf{x}) = \delta^{(3)}(\mathbf{x}) \;\Rightarrow\; \Phi(\mathbf{x}) = -\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{e^{+i\mathbf{k}\cdot\mathbf{x}}}{k^2+m^2},$$

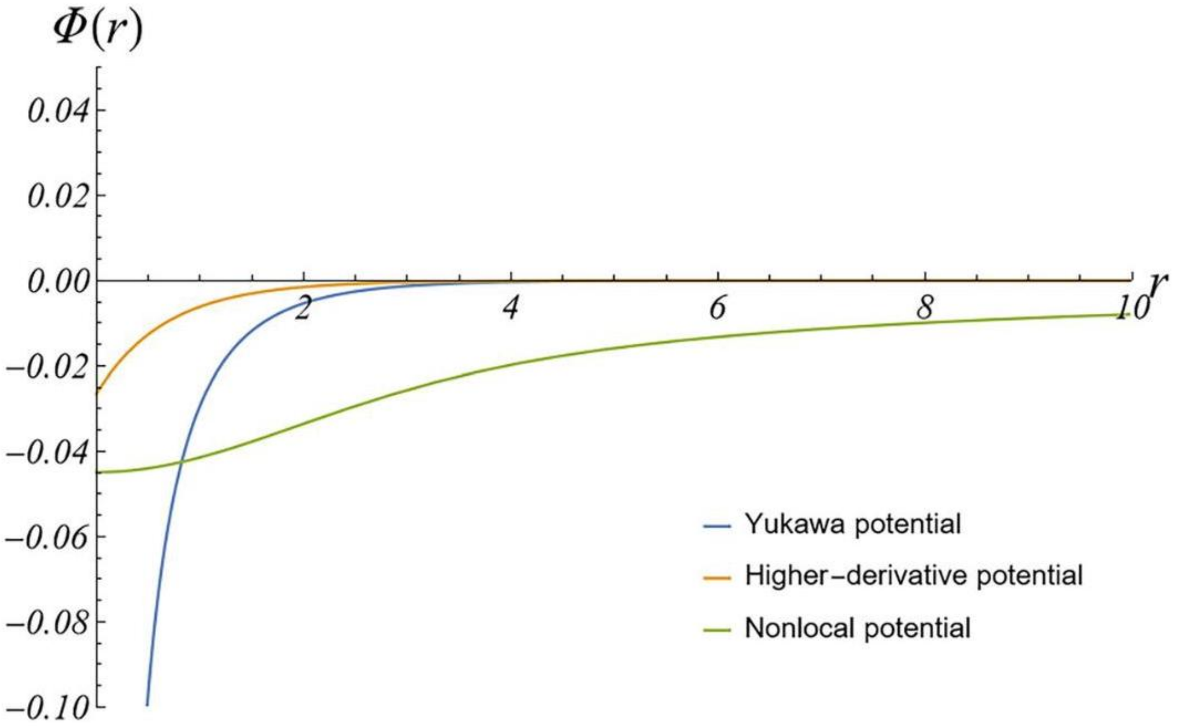


Fig. 4$\Phi(r)$ for the three cases described in the text taking $l_* = 1, m = m_1 = 1$, and $m_2 = 2$. In blue the standard Yukawa potential, in orange the higher-derivative (four-derivative) case and in green the nonlocal (Wataghin form factor) case

图 4$\Phi(r)$ 为文中描述的三种情形，取值为 $l_* = 1, m = m_1 = 1$ 和 $m_2 = 2$。蓝色为标准汤川势，橙色为高阶导数 (四阶导数) 情形，绿色为非局域 (Wataghin 形状因子) 情形

whose value is given by

其值由下式给出

$$\Phi(r) = -\frac{e^{-mr}}{4\pi r}, \tag{118}$$

which is divergent in the limit $r = |\mathbf{x}| \to 0$. When the mass parameter $m = 0$, then we have the standard Newtonian potential falling off as $1/r$.

该式在 $r = |\mathbf{x}| \to 0$ 极限下发散。当质量参数为 $m = 0$ 时，我们得到标准牛顿势，其衰减规律为 $1/r$。

- Quartic in derivatives local HD form factor. In this case, $\gamma(\square) = -(\square - m_1^2)\,(\square - m_2^2)$ and

- 导数四次方的局域高阶导数形状因子。该情形下，$\gamma(\square) = -(\square - m_1^2)\,(\square - m_2^2)$，且

$$-\left(\Box-m_1^2\right)\left(\Box-m_2^2\right)\Phi\left(x\right)=\delta^{(3)}\left(\mathbf{x}\right)\Rightarrow\Phi\left(\mathbf{x}\right)$$

$$=-\int\frac{d^3\mathbf{k}}{\left(2\pi\right)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\left(k^2+m_1^2\right)\left(k^2+m_2^2\right)}. \tag{119}$$

Splitting the integral as

将积分拆分为

$$\frac{1}{\left(k^2+m_1^2\right)\left(k^2+m_2^2\right)}=\frac{1}{m_1^2-m_2^2}\left[\frac{1}{k^2+m_2^2}-\frac{1}{k^2+m_1^2}\right],$$

we can make use of the results of the Yukawa potential to write

我们可以利用汤川势的结果得到

$$\Phi\left(r\right)=\frac{1}{m_1^2-m_2^2}\frac{e^{-m_1r}-e^{-m_2r}}{4\pi r}. \tag{120}$$

When taking the limit $r\to 0,\Phi\left(r\right)$ remains finite,

当取 $r\to 0,\Phi\left(r\right)$ 极限时结果仍保持有限,

$$\Phi\left(0\right)=-\frac{1}{4\pi\left(m_1+m_2\right)}. \tag{121}$$

- Wataghin form factor. In this case, we choose $\gamma\left(\Box\right)=e^{-l_*^2\Box}\Box$, so that the potential is given by

- Wataghin 形状因子。在该情形下，我们选择 $\gamma\left(\Box\right)=e^{-l_*^2\Box}\Box$ ，因此势由下式给出

$$e^{-l_*^2\Box}\Box\Phi\left(\mathbf{x}\right)=\delta^{(3)}\left(\mathbf{x}\right)\;\Rightarrow\;\Phi\left(\mathbf{x}\right)=\int\frac{d^3\mathbf{k}}{\left(2\pi\right)^3}\frac{e^{+i\mathbf{k}\cdot\mathbf{x}-l_*^2k^2}}{k^2}. \tag{122}$$

The final result is

最终结果为

$$\Phi\left(r\right)=-\frac{1}{4\pi r}\,\mathrm{erf}\left(\frac{r}{2l_*}\right), \tag{123}$$

where erf is the error function. When taking the limit $r\to 0$, one finds

其中 erf 为误差函数。当取 $r\to 0$ 极限时，可以得到

$$\Phi\left(r\to 0\right)=-\frac{1}{4\pi^{3/2}l_*}, \tag{124}$$

which is finite.

结果为有限值。

In conclusion, using nonlocal form factors allows one to remove the classical divergences of gravity by smearing the gravitational source, in the same way Wataghin [14] gave a finite radius to point-like particles using nonlocal operators.

综上所述，利用非局部形状因子可以通过涂抹引力源消除引力的经典发散，这与 Wataghin[14] 利用非局部算子给类点粒子赋予有限半径的方法如出一辙。

Some comments about these results are in order here. First, we have not derived the Yukawa (Newtonian) potential from the full equations of motion using the diffusion method but we assumed the modified Poisson equation (117). Second, the analysis was done for the non-relativistic linear Yukawa potential, without non-linearities and using the Fourier transform. However, this procedure might not give all solutions, since a hidden condition is that Φ fall off at spatial infinity. Third, in section "Ricci-Flat Spacetimes" we showed that Ricci-flat spacetimes are also solutions of the equations of motion of NLG, which means that there do exist singular black holes such as those described by Schwarzschild and Kerr metrics.

在此我们对这些结果做几点说明。第一，我们并未利用扩散方法从完整运动方程推导出汤川 (牛顿) 势，而是假设了修正泊松方程 (117) 成立。第二，我们的分析是针对非相对论线性汤川势开展的，没有考虑非线性项，且使用了傅里叶变换。但这套流程可能无法得到所有解，因为它隐含了 Φ 在空间无穷远处衰减的条件。第三，我们在“里奇平坦时空”一节中已经证明，里奇平坦时空同样是非局部引力理论 (NLG) 运动方程的解，这说明该理论中确实存在史瓦西和克尔度规描述的奇异黑洞。

There are two ways in which NLG can approach the singularity problem based on which type of nonlocal theory (70) we adopt.

根据我们采用的非局部理论类型 (70)，NLG 处理奇点问题有两种不同方式。

- If we omit the Riemann-Riemann term and set $\gamma_4(\Box) = 0$, as we did in sections "Ricci-Flat Spacetimes" and "Stability", then the results of Ricci-flat spacetimes apply and the singularities of the Schwarzschild and Kerr black holes are transferred to the nonlocal theory. Therefore, this nonlocal gravitational theory is not capable to tame the singularities of GR at the classical level. Fortunately, conformal invariance solves the problem at the quantum level [50]. The status of Ricci-flat solutions as physical is also not completely established. In fact, one can argue that, since astronomical black holes are formed by gravitational collapse, there must be matter inside the event horizon, so that vacuum solutions to Einstein' s equations are not valid at the singularity $r = 0$. Thus, as is well known in GR, Schwarzschild's metric (104) does not describe all the spacetime but only a patch of it, and the matter distribution associated with Schwarzschild should be computed solving Einstein's equations in the sense of distribution. This has been done for GR [51, 52] but not yet in NLG, so that we do not really know that the singularity of the metric consistently matches a delta-like matter distribution at the centre of the black hole.

- 若我们去掉黎曼-黎曼项并设为 $\gamma_4(\Box) = 0$，就像我们在“里奇平坦时空”和“稳定性”两节中所做的那样，那么里奇平坦时空的结论仍然成立，史瓦西黑洞和克尔黑洞的奇点也会直接延续到非局部理论中。因此这类非局部引力理论无法在经典层面驯服广义相对论 (GR) 的奇点。幸运的是，共形不变性可以在量子层面解决该问题 [50]。目前里奇平坦解的物理地位也未完全确立。事实上我们可以说，由于天文黑洞是引力坍缩形成的，事件视界内部必然存在物质，因此爱因斯坦方程的真空解在奇点 $r = 0$ 处并不成立。正如广义相对论中众所周知的，史瓦西度规 (104) 无法描述整个时空，只能描述其中一片区域，对应史瓦西解的物质分布需要在分布意义下求解爱因斯坦方程得到。这一工作已经在广义相对论中完成 [51, 52]，但尚未在 NLG 中实现，因此我们并不确定度规奇点能否与黑洞中心的类 δ 物质分布自洽匹配。

- Setting $\gamma_4(\Box) \neq 0$, the Ricci-flat results are no longer valid since in the equations of motion we have terms involving the Riemann tensor, which, in general, is non-vanishing even for Ricci-flat spacetimes. In this case, nonlocality can make the singularities disappear already at the classical level [3, 53, 54] . Indeed, the higher-derivative and nonlocal operators studied in this review smooth out point-like singularities [55-57], as seen in equations (120) and (123). Also, it has been proven that the presence of a Riemann-Riemann term in the Lagrangian forbids Schwarzschild-type singularities with $\Phi(r) \sim 1/r^{\alpha}$ for $\alpha > 1$ [57, 58] . However, a problem of the nonlocal theory with the Riemann-Riemann tensor terms is that, in general, it is difficult to find exact solutions.

- 设 $\gamma_4(\Box) \neq 0$ 时，里奇平坦的结论不再成立，因为运动方程中会出现含黎曼张量的项，而即使对于里奇平坦时空，黎曼张量一般也不为零。在这种情况下，非局域性可以在经典层面就让奇点消失 [3, 53, 54] 。实际上，本综述中研究的高阶导数非局部算子可以抹除类点奇点 [55-57]，这一点可以从方程 (120) 和 (123) 中看出。此外已经证明，拉格朗日中存在黎曼-黎曼项时，对于 $\alpha > 1$ [57, 58] 会禁止存在满足 $\Phi(r) \sim 1/r^{\alpha}$ 的史瓦西型奇点。不过，带有黎曼-黎曼张量项的非局部理论存在一个普遍问题: 很难找到精确解。

Nonlocal Quantum Scalar Field Theory

非局域量子标量场论

Having studied the classical theory of nonlocal interactions (or nonlocal kinetic terms, which is the same as we saw in section ”Nonlocality”), we now focus on how we quantize this theory and how unitarity is preserved at the quantum level. First of all, we review some basics about power counting renormalization as well as the derivation of the unitarity bound. Subsequently, we analyse the way nonlocal form factors alter the prescription of local theories to deal with momentum integrals of the quantum theory and how we can employ Efimov’s analytic continuation to build a meaningful quantum theory. To conclude, we introduce the Cutkosky rules used to verify the perturbative unitarity of the nonlocal scalar theory.

在研究了非局域相互作用 (即非局域动能项，正如我们在“非局域性”小节中看到的，二者等价) 之后，我们现在聚焦于如何对该理论进行量子化，以及如何在量子层面保持幺正性。首先，我们回顾幂次计数可重整性的基础内容以及幺正界的推导。随后，我们分析非局域形状因子如何修改局域理论处理量子理论动量积分的规则，以及我们如何利用埃菲莫夫解析延拓构建一个自洽的量子理论。最后，我们介绍用于验证非局域标量理论微扰幺正性的库茨科夫斯基规则。

Power-Counting Renormalizability

幂次计数可重整性

Renormalization is a property of a quantum theory that tells us whether our theory may calculate physical quantities, namely if it is predictive. In practice, the renormalization procedure is related to the way we manage to deal with the infinities that can appear in momentum integrals in a quantum theory. There are many ways in which we can regularize a theory, i.e., express these infinities, such as the cutoff scheme, dimensional regularization, and so on. Once the theory is regularized, one can check whether it can be renormalized. In this section, we approach the renormalizability of a theory via the so-called power-counting renormalizability, which is a useful criterion to know how badly the scattering amplitudes of the theory will be divergent.

重整化是量子理论的一种性质，它告诉我们理论能否计算物理量，即理论是否具有可预言性。实际上，重整化过程与我们处理量子理论中动量积分可能出现的无穷大的方式有关。我们可以通过多种方法正规化理论，也就是分离出这些无穷大，例如截断方案、维数正规化等等。理论正规化后，就可以检验它是否可被重整化。本节我们通过所谓的幂次计数可重整性来研究理论的可重整性，这是一个判断理论散射振幅发散程度的有效判据。

First, let us review how this criterion works for the local scalar field theory. In general, the Lagrangian of these theories can be written as

首先，我们先回顾该判据在局域标量场论中是如何成立的。一般来说，这类理论的拉格朗日量可以写为

$$\mathcal{L}_\phi = \sum_n \lambda_n \mathcal{O}^{(n)}(\phi) \tag{125}$$

where $\mathcal{O}^{(n)}$ denotes an interacting term for the scalar field ϕ with energy dimension n and λ_n are the bare coupling constants. After renormalization, these couplings acquire a dependence on the energy scale of the given process and are not constant in general.

其中 $\mathcal{O}^{(n)}$ 表示标量场 ϕ 的相互作用项，量纲为能量的 n 次，λ_n 是裸耦合常数。重整化后，这些耦合会依赖于对应过程的能标，通常不再是常数。

Depending on the way scalar fields interact in $\mathcal{O}^{(n)}$ the energy dimensionality of the coupling λ_n will change. In particular, if D is the topological dimension of the spacetime, we may distinguish three different cases.

$\mathcal{O}^{(n)}$ 中标量场的相互作用方式不同，耦合 λ_n 的能量量纲也会随之改变。具体来说，若 D 是时空的拓扑维数，我们可以区分出三种不同情况。

- $n < D$. In this case, $[\lambda_n] > 0$, and the coupling will decrease as we increase the energy, so that this kind of operator is called relevant, since it becomes important in the IR regime.

- $n < D$ 。在这种情况下，$[\lambda_n] > 0$，耦合会随能量升高而减小，因此这类算子被称为相关算子，因为它在红外区域会变得重要。

- $n = D$. In this case, $[\lambda_n] = 0$, and the coupling will not depend on the energy scale, so that this kind of operator is called marginal.

> - $n = D$ 。在这种情况下，$[\lambda_n] = 0$ ，耦合不依赖于能标，因此这类算子被称为临界算子。

- $n > D$. In this case, $[\lambda_n] < 0$, and the coupling will grow as we go to higher and higher energies, so that this kind of operator is called irrelevant, since it becomes important in the UV regime. These constitute the non-renormalizable couplings of a theory (In $D = 4$ dimensions, the allowed renormalizable interactions at the Lagrangian level are ϕ^2, ϕ^3 , and ϕ^4 , since interactions of the kind ϕ^n for $n \geq 5$ are non-renormalizable. The standard model of particle physics is built out of field operators of maximal energy dimension 4.).

> - $n > D$ 。在这种情况下，$[\lambda_n] < 0$ ，耦合会随能量升高而增大，因此这类算子被称为不相关算子，因为它在紫外区域才会变得重要。这类算子对应理论中的不可重整耦合 (在 $D = 4$ 维时空中，拉格朗日层面允许的可重整相互作用是 ϕ^2, ϕ^3 和 ϕ^4 ，因为对于 $n \geq 5$ ，形式为 ϕ^n 的相互作用都是不可重整的。粒子物理的标准模型由最大能量量纲为 4 的场算子构成。)。

Once identified the type of operators that our scalar theory might have, one applies the renormalization procedure, that allows us to reabsorb the divergences of scattering amplitudes. In particular, the scattering amplitudes $\mathcal{M}$ involving some of the previous operators diverge. However, one can reabsorb these infinities order by order by including higher-order operators at the Lagrangian level, so that the divergent part of the amplitude is cancelled out. In the case of irrelevant operators, one needs to add an infinite number of higher-order operators, which results in a non-renormalizable theory. In this context, one says that the theory is power-counting renormalizable if the interactions satisfy $[\lambda_n] \geq 0$.

> 确定标量理论可能包含的算子类型后，就可以应用重整化过程，该过程可以让我们重新吸收散射振幅的发散。具体来说，包含上述算子的散射振幅 $\mathcal{M}$ 会发散。但我们可以通过在拉格朗日量层面引入高阶算子，逐阶重新吸收这些无穷大，抵消振幅的发散部分。对于不相关算子，我们需要引入无穷多个高阶算子，最终得到一个不可重整理论。在此框架下，若相互作用满足条件 $[\lambda_n] \geq 0$ ，则称理论是幂次计数可重整的。

In the nonlocal case, one must be more careful with these arguments, since there are infinitely many interactions that involve derivatives. We introduce the superficial degree of divergence $\omega(\mathcal{F})$ of a Feynman diagram $\mathcal{F}$ as a criterion to classify a given theory as renormalizable or non-renormalizable. This number characterizes a particular Feynman diagram and it encodes the energy dependence of the scattering amplitude associated with it when computing the corresponding integrals up to a given cut-off scale Λ that will be taken to infinity:

> 讨论非局域情形时，我们需要更谨慎地对待上述论证，因为非局域理论中存在无穷多包含导数的相互作用。我们引入费曼图 $\mathcal{F}$ 的表面发散度 $\omega(\mathcal{F})$ 作为判据，将给定理论归类为可重整或不可重整。这个量表征了特定费曼图的性质，它编码了对应散射振幅在计算到给定截断能标 Λ (最终会取无穷大) 时对能量的依赖关系:

$$\mathcal{M} \propto \Lambda^{\omega(\mathcal{F})}. \tag{126}$$

From this expression, one concludes that:

从该表达式可以得到结论:

- $\omega(\mathcal{F}) < 0 \Rightarrow \mathcal{F}$ is superficially convergent.

  - $\omega(\mathcal{F}) < 0 \Rightarrow \mathcal{F}$ 是表面收敛的。

- $\omega(\mathcal{F}) = 0 \Rightarrow \mathcal{F}$ diverges logarithmically.

  - $\omega(\mathcal{F}) = 0 \Rightarrow \mathcal{F}$ 呈对数发散。

- $\omega(\mathcal{F}) > 0 \Rightarrow \mathcal{F}$ is superficially divergent.

  - $\omega(\mathcal{F}) > 0 \Rightarrow \mathcal{F}$ 是表面发散的。

Actually, the power-counting renormalization is not a necessary condition for the renormalizability of the theory itself. For instance, there are finite scattering amplitudes whose superficial degree of divergence is positive, such as the one-loop 4-photon diagram in QED that cancels out by Furry's theorem despite $\omega(\mathcal{F}) > 0$ [59].

实际上，幂计数可重整化并不是理论本身可重整化的必要条件。例如，存在表面发散度为正但散射振幅有限的情况: 量子电动力学中的单圈四光子图就是如此，尽管存在 $\omega(\mathcal{F}) > 0$ ，仍可通过富里定理抵消发散 [59]。

Therefore, we may use this quantity to qualitatively classify a Feynman diagram as divergent or not, but always keeping in mind that there could be some symmetry mechanism in the system that ameliorates the divergent behaviour of a given scattering amplitude. In contrast, power-counting renormalizability is a sufficient condition for renormalizability, so that, if a theory is found to be power-counting renormalizable, then explicit calculations of scattering amplitudes will only confirm this result.

因此，我们可以用这个量对费曼图是否发散做定性分类，但需要始终注意，体系中可能存在某种对称性机制，改善给定散射振幅的发散行为。反之，幂计数可重整化是可重整化的充分条件: 如果一个理论满足幂计数可重整化，那么散射振幅的显式计算只会验证这一结论。

Unitarity

幺正性

In a QFT, one of the most important quantities is the so-called scattering amplitude, often denoted by $\mathcal{M}$. The square of this object, $|\mathcal{M}|^2$, is used to calculate cross-sections of a particular decay and characterizes the probability with which this process can take place. Therefore, the calculation of $|\mathcal{M}|^2$ can be directly compared with experimental data to validate or not a theory.

在量子场论中，最重要的物理量之一就是所谓的散射振幅，通常记作 $\mathcal{M}$ 。该量的模平方 $|\mathcal{M}|^2$ 可用于计算特定衰变过程的截面，描述该过程发生的概率。因此，$|\mathcal{M}|^2$ 的计算结果可以直接和实验数据对比，以此验证理论是否正确。

From a classical point of view, we briefly mentioned in section "Unstable Modes and Ostrogradski's Theorem" that the existence of higher-order derivatives in a theory gives rise to unstable modes that, in turn, lead to a spontaneous decay of the vacuum. In this sense, we say that unitarity is broken and, since our theory cannot reproduce a physical universe, it must be ruled out. In the quantum regime, this condition of unitarity can be easily encoded in the S-matrix, defined as the matrix $\hat{S}$ that connects the initial state a and the final state b in a particular decay. Mathematically, it can be written as

从经典视角来看，我们在“不稳定模式与奥斯特罗格拉德斯基定理”一节中简要提到，理论中存在高阶导数会催生不稳定模式，进而导致真空自发衰变。在这种情况下，我们认为幺正性被破坏；由于这类理论无法重现我们所处的物理宇宙，因此必须被排除。在量子层面，幺正性的要求可以很方便地用 S 矩阵来表述:S 矩阵是连接特定衰变过程中初态 a 和末态 b 的矩阵 $\hat{S}$ ，其数学形式可以写为

$$\mathcal{S}_{a\to b} = \langle b|\hat{S}|a\rangle. \tag{127}$$

By conservation of probability, one has that the S-matrix is a unitary matrix that satisfies

根据概率守恒，S 矩阵是满足下述条件的幺正矩阵

$$\hat{S}^\dagger \hat{S} = \mathbb{1}. \tag{128}$$

It is usually convenient to split the S-matrix into 2 parts: one that describes the free theory in which no interaction is taken into account and another whose main purpose is to describe interactions. One writes this splitting as

我们通常可以方便地将 S 矩阵拆分为两部分: 一部分描述不考虑相互作用的自由理论，另一部分用于描述相互作用。拆分形式写作

$$\hat{S} = \mathbb{1} + i\widehat{T} \tag{129}$$

where the matrix $\widehat{T}$ has been introduced.

其中我们引入了矩阵 $\widehat{T}$ 。

Optical Theorem

光学定理

Inserting the splitting (129) into the unitarity condition (128), one obtains

将拆分式 (129) 代入幺正性条件 (128)，可得

$$-i\left(\widehat{T}-\widehat{T}^{\dagger}\right)=\widehat{T}^{\dagger}\widehat{T}. \tag{130}$$

We can act on this expression with an initial state $|i\rangle$ and a final state $|f\rangle$ and define the amplitudes $\mathcal{T}_{fi}=\langle f|\widehat{T}|i\rangle$. Using this notation, we have that

我们可以将初态 $|i\rangle$ 和末态 $|f\rangle$ 代入该表达式，并定义振幅 $\mathcal{T}_{fi}=\langle f|\widehat{T}|i\rangle$ 。使用该记号，我们得到

$$-i\left(\langle f|\widehat{T}|i\rangle-\langle f|\widehat{T}^{\dagger}|i\rangle\right)=-i\left(\mathcal{T}_{fi}-\mathcal{T}_{if}^{*}\right)=\left\langle f\left|\widehat{T}^{\dagger}\widehat{T}\right|i\right\rangle. \tag{131}$$

Considering a theory invariant under the reflection $x^{\mu}\to -x^{\mu}$, one has that $\widehat{T}$ is symmetric, and inserting a completeness relation in between $\widehat{T}$ and $\widehat{T}^{\dagger}$ in the righthand side of (131), one obtains

考虑在反射 $x^{\mu}\to -x^{\mu}$ 下不变的理论，可知 $\widehat{T}$ 是对称的；在 (131) 右侧的 $\widehat{T}$ 和 $\widehat{T}^{\dagger}$ 之间插入完备性关系，可得

$$-i\left(\mathcal{T}_{fi}-\mathcal{T}_{fi}^{*}\right)=2\,\mathrm{Im}\,\mathcal{T}_{fi}=\sum_{n}\mathcal{T}_{nf}^{*}\mathcal{T}_{ni}. \tag{132}$$

This result is known as the optical theorem and, as we will see shortly, it will be of crucial importance in order to test the unitarity of a nonlocal theory. Besides, in the particular case of the forward scattering, i.e., $i=f$, one finds that in a theory with no ghosts [59]

该结果即为光学定理, 我们很快就会看到, 它对检验非局域理论的幺正性至关重要。此外, 在 forward scattering(正向散射) 的特殊情况，即 $i=f$ 下，发现在无鬼的理论中有 [59]

$$2\,\mathrm{Im}\,T_{ii}=\sum_{n}|\mathcal{T}_{ni}|^{2}>0\;\Rightarrow\;\mathrm{Im}\,T_{ii}>0. \tag{133}$$

Furthermore, if one factorizes out the momentum dependence that always appears in these matrix elements and instead works in terms of the scattering amplitude $\mathcal{M}$ defined as

此外, 如果我们将这些矩阵元中始终存在的动量依赖因子分解出来, 改用散射振幅 $\mathcal{M}$ 进行计算，该散射振幅定义为

$$\mathcal{T}_{ij}=(2\pi)^{4}\delta^{(4)}\left(\mathrm{P}_{T}\right)\mathcal{M}_{ij}, \tag{134}$$

one has that (133) becomes

可得到 (133) 变为

$$|\mathcal{M}_{ii}|\geq\mathrm{Im}\,\mathcal{M}_{ii}=\frac{1}{2}\sum_{n}|\mathcal{M}_{ni}|^{2}\geq\frac{1}{2}|\mathcal{M}_{ii}|^{2}\Rightarrow|\mathcal{M}_{ii}|\leq 2. \tag{135}$$

In addition, using this result for $|\mathcal{M}_{ii}|$ and considering a general state j , one has that

另外，将该结果用于 $|\mathcal{M}_{ii}|$ 并考虑一般态 j ，可得

$$2 \geq |\mathcal{M}_{ii}| \geq \operatorname{Im}\mathcal{M}_{ii} = \frac{1}{2}\sum_{n}|\mathcal{M}_{ni}|^2 \geq \frac{1}{2}|\mathcal{M}_{ij}|^2 \Rightarrow |\mathcal{M}_{ij}| \leq 2.$$

(136)

This unitarity bound must be satisfied by any quantum field theory that preserves unitarity. In section ”Tree-Level Unitarity”, we will show that, indeed, this unitarity bound is satisfied in NLQG.

任何保持幺正性的量子场论都必须满足该幺正界。在“树级幺正性”一节中我们将会证明，非局域量子引力 (NLQG) 确实满足该幺正界。

Lorentzian and Euclidean Momentum Space

洛伦兹动量空间与欧几里得动量空间

As briefly mentioned in section ”Summary of Form Factors”, we can define a quantum field theory either in Lorentzian momentum space or in Euclidean momentum space. An analytic continuation $k_E = -ik^0$ of the energy component of the momenta, making it purely imaginary, connects the two formulations by a Wick rotation. Since the propagator usually has poles in k^0 , which are identified with particle modes of the theory, one extends the domain of k^0 to the complex plane using an analytic continuation and uses the Feynman prescription $i\varepsilon$ to displace these poles from the real axis. As a result of this displacement in the complex plane, one may use Cauchy’s theorem to calculate the integrals that usually appear in any quantum theory, namely,

正如“形状因子总结”一节中简要提及的，我们既可以在洛伦兹动量空间中定义量子场论，也可以在欧几里得动量空间中定义它。对动量的能量分量做解析延拓 $k_E = -ik^0$ ，使其成为纯虚数，即可通过威克转动连接这两种表述。由于传播子通常在 k^0 存在极点，这些极点对应理论的粒子模式，因此我们通过解析延拓将 k^0 的定义域扩展至复平面，并采用费曼规则 $i\varepsilon$ 将这些极点移出实轴。经过复平面上的极点移位后，我们就可以利用柯西定理计算任意量子理论中通常都会出现的积分，即:

$$I = \int_{-\infty}^{+\infty} dk^0 f\left(k^0\right). \tag{137}$$

In this way, one relates both momenta prescription to characterize the same theory and one can calculate quantities in Euclidean momentum space and then apply an inverse (clockwise) Wick rotation to return to the theory in Lorentzian momentum. This is the standard procedure when dealing with local scalar field theories.

通过这种方式，我们将两种动量表述关联起来描述同一个理论: 可以先在欧几里得动量空间中计算物理量，再通过逆 (顺时针) 威克转动回到洛伦兹动量下的理论，这是处理局域标量场论的标准流程。

However, in nonlocal field theories this equivalence is not available since the contributions of the arcs at infinity, in general, do not vanish because of the additional momentum dependence in the propagator. In fact, most nonlocal propagators diverge in some quadrant of the complex plane when going to infinity.

> 但在非局域场论中，这种等价性不成立: 由于传播子存在额外的动量依赖，无穷远处圆弧的贡献一般不会消失，实际上大多数非局域传播子在复平面的某个象限趋向无穷时都会发散。

As showed in Fig. 2, it is desirable to define the nonlocal theory in Euclidean momenta since the UV behaviour of the nonlocal form factors of our interest is divergent, the propagator is highly suppressed in this regime and, as a consequence, the potential divergences of the quantum theory are reduced. Since we do not have Wick rotation at our disposal to convert to Lorentzian signature, we need to implement an alternative procedure called Efimov analytic continuation. Before we introduce this prescription, we talk about an important feature of nonlocal theories that allows us to redefine our fields without inducing observable consequences.

> 如图 2 所示，我们更适合在欧几里得动量中定义非局域理论: 我们关注的非局域形状因子其紫外行为本身发散，而在欧几里得动量框架下传播子在该区域会被强烈压低，因此量子理论的潜在发散会得到缓解。由于我们无法通过威克转动转换到洛伦兹符号，因此需要采用一种名为叶菲莫夫解析延拓的替代流程。在介绍这一规则之前，我们先讨论非局域理论的一个重要特性: 它允许我们重新定义场，且不会产生可观测的影响。

Nonlocal Field Redefinition

非局域场重定义

One aspect of nonlocal field theories that we have not approached yet is a possible nonlocal redefinition of the fields, so that in some way we may get an equivalent local theory. To be more precise, given the nonlocal theory

> 非局域场论中我们尚未讨论的一个方面是可能存在的场的非局域重定义，借此我们或许能得到一个等价的局域理论。更准确地说，给定非局域理论

$$\mathcal{L} = \frac{1}{2}\phi\gamma(\Box)\phi - V(\phi), \tag{138}$$

is it possible to redefine our field as $\phi \to \widetilde{\phi}(\phi)$ such that the theory becomes local without modifying the physics? This is in general possible under some particular assumptions. However, to show it explicitly we make use of the following form factor:

> 我们能否将场重定义为 $\phi \to \widetilde{\phi}(\phi)$，使得理论在不改变物理内容的前提下变为定域理论？一般而言，这在某些特定假设下是可行的。不过为了明确展示这一点，我们使用如下形状因子:

$$\gamma(\Box) = e^{\mathrm{H}(\Box)}(\Box - m^2). \tag{139}$$

There exist equivalence theorems [60, 61] proving that, under nonlinear local field redefinition at the Lagrangian level, one obtains the same scattering amplitudes than the original theory; namely, under a transformation of the form

已有等价定理 [60, 61] 证明: 在拉格朗日量层面进行非线性定域场重定义，得到的理论会给出和原理论相同的散射振幅；也就是说，在如下形式的变换下

$$\widetilde{\phi}(\phi)=\phi+f(\phi,\nabla\phi), \tag{140}$$

the observable quantities remain invariant. However, although it was shown that this equivalence also holds for certain nonlocal redefinitions [62], in general a nonlocal field redefinition might not lead to an equivalent theory. Here we focus only on non-singular and nonzero nonlocal form factors, so that a field redefinition

可观测量保持不变。然而，尽管已有研究证明这种等价性对特定非局域重定义也成立 [62]，一般情况下非局域场重定义并不一定能得到等价理论。本文我们仅讨论非奇异、非零的非局域形状因子，因此场重定义

$$\widetilde{\phi}(\phi)=e^{\frac{1}{2}\mathrm{H}(\Box)}\phi, \tag{141}$$

does leave the particle spectrum invariant, since no new poles appear in the propagator. Therefore such a nonlocal transformation leads to an equivalent theory [62].

确实能保持粒子谱不变，因为传播子中不会出现新的极点。因此这类非局域变换可以得到等价理论 [62]。

Under the redefinition (141), (138) transforms into

在重定义 (141) 下，(138) 变换为

$$\mathcal{L}=\frac{1}{2}\widetilde{\phi}\left(\Box-m^2\right)\widetilde{\phi}-V\left[e^{-\frac{1}{2}\mathrm{H}(\Box)}\widetilde{\phi}\right], \tag{142}$$

so that in the free field theory ($V=0$) the nonlocal redefinition (141) leads to a local scalar field theory. In general, we can always transfer all the nonlocality to the potential, which usually is a polynomial of third or higher order, leading to nonlocal interactions and vertices. As a result of this, we may use the standard propagator of the local theory given by

因此在自由场理论 ($V=0$) 中，非局部重定义 (141) 可导出一个局部标量场理论。一般而言，我们总能将所有非 locality 转移到势项中，势项通常是三阶及以上的多项式，由此产生非局部相互作用和顶点。基于此，我们可以使用局部理论给出的标准传播子，即

$$G\left(k^2\right)=\frac{1}{-k^2-m^2+i\varepsilon}. \tag{143}$$

Unitarity

幺正性

As we said, we cannot apply a Wick rotation to establish the equivalence of the quantum theory expressed in Euclidean and Lorentzian momentum space. However, Efimov worked out a consistent way to identify both formulations [63] which is based on an analytic continuation to the complex plane of the time-like component of both the internal and external momenta in a Feynman diagram. In this prescription, one carries out the explicit calculations integrating the time-like component of the momentum along the imaginary axis and afterwards, the external momentum is analytically continued back to its real value. The main difference with respect to the traditional Wick rotation is that the way to return to the real values of the external momenta is not achieved by something looking like a rigid rotation but by a specific and more complicated deformation of the integration contour in the complex plane.

如前文所述，我们无法通过威克旋转证明欧几里得动量空间与洛伦兹动量空间下的量子理论是等价的。但叶菲莫夫找到了一种能够兼容两种表述的一致方案 [63]: 该方案基于对费曼图中内、外动量类时分量向复平面的解析延拓。在此方案中，人们沿虚轴对动量的类时分量做显式积分计算，之后再将外动量解析延拓回其真实值。与传统威克旋转的核心区别在于，将外动量还原为实值的过程并非通过类似刚性旋转的操作完成，而是通过对复平面积分围道做特殊、更复杂的形变实现。

The optical theorem stated in section ”Optical Theorem” can be generalized via the Cutkosky rules [64] which can be applied to the nonlocal Lagrangian (142) in order to prove the unitarity of the theory. These Cutkosky rules are based on the possibility to cut a Feynman diagram into two pieces so that we may decompose the full diagram into the sum of intermediate states. As we have showed in section ”Optical Theorem”, to prove unitarity we need to focus on the imaginary part of the scattering amplitude $\mathcal{M}$. In this context, the Cutkosky rules state that, if the theory is unitary, then we can replace the internal propagators by a delta function [64]:

” 光学定理” 一节给出的光学定理可以通过卡特斯基规则 [64] 推广，该规则可应用于非局域拉格朗日量 (142)，以证明该理论的幺正性。卡特斯基规则的核心是将费曼图切割为两部分，从而将完整图分解为中间态的求和。正如我们在” 光学定理” 一节中展示的，证明幺正性需要聚焦于散射振幅 $\mathcal{M}$ 的虚部。在此框架下，卡特斯基规则指出: 若理论是幺正的，我们就可以将内传播子替换为 δ 函数 [64]:

$$\frac{i}{k^2+m^2-i\varepsilon} \to 2\pi i\delta\left(k^0\right)\delta\left(k^2+m^2\right). \tag{144}$$

In conclusion, to prove perturbative unitarity of the nonlocal theory, one can calculate its scattering amplitudes $\mathcal{M}$ explicitly and show that the imaginary part is given by the Cutkosky rules. Since these rules hold when the theory is unitary, one shows that the nonlocal theory does not violate unitarity [65].

综上，要证明非局域理论的微扰幺正性，可以显式计算其散射振幅 $\mathcal{M}$ ，并证明其虚部满足卡特斯基规则。由于该规则在理论幺正时成立，因此可以证明非局域理论不违反幺正性 [65]。

Nonlocal Quantum Gravity

非局域量子引力

The last part of this chapter is dedicated to show how nonlocal gravity solves the obstacles that Einstein's gravity poses when one tries to quantize it. First of all, we expose the problems that GR has in the context of renormalization as well as the way Stelle's theory overcomes them. Then we tackle the problem of ghosts. We have already seen in section "Diffusion Method for Nonlocal Gravity" the classical instabilities of this modified gravity caused by the appearance of kinetic terms in the linearized action with the wrong sign. Here we show how Stelle's gravity violates the unitarity bound derived in section "Optical Theorem", as well as how nonlocality provides an answer to the unitarity problem.

本章最后一部分旨在说明非局域引力如何解决爱因斯坦引力在量子化过程中遇到的障碍。首先，我们阐述广义相对论在重整化背景下存在的问题，以及斯泰勒理论解决这些问题的方法。随后我们讨论鬼场问题: 我们已经在“非局域引力的扩散方法”一节中介绍了这类修正引力因线性化作用量中出现符号错误的动力学项而产生的经典不稳定性，本文将说明斯泰勒引力如何违反“光学定理”一节中推导得到的幺正性界，以及非局域性如何为幺正性问题提供解决方案。

Secondly, we can see that no breaking of causality is induced in this theory by looking at Shapiro's time delay. We also study the superficial degree of divergence $\omega(\mathcal{F})$ of the theory and see that $\omega(\mathcal{F})$ is positive in the one-loop case so that we focus on this particular setup. After showing how to achieve renormalizability for the nonlocal theory, we highlight the asymptotic freedom that the theory exhibits.

其次，我们可以通过研究夏皮罗时滞证明该理论不会引发因果性破缺。我们还研究了该理论的表观发散度 $\omega(\mathcal{F})$，发现在单圈情况下 $\omega(\mathcal{F})$ 为正，因此我们聚焦于这一特定框架。在展示非局域理论如何实现可重整化性之后，我们重点介绍该理论展现出的渐近自由性质。

In order to quantize the gravitational field, we split the metric field into a background field that will be chosen to be flat spacetime, and a perturbation that is identified with the graviton. Under the decomposition (98), one rewrites Einstein's action (1) in terms of the perturbation $h_{\mu\nu}$ to obtain a canonical kinetic term proportional to $(\partial h)^2$, as well as higher-order interactions of the graviton. The resulting Lagrangian contains graviton interactions with coupling constants whose energy dimension is negative, leading to a non-renormalizable operator studied through power-counting arguments.

为了对引力场进行量子化，我们将度规场拆分为两部分: 选作平坦时空的背景场，以及对应引力子的微扰项。根据分解式 (98)，我们可以将爱因斯坦作用量 (1) 用微扰项 $h_{\mu\nu}$ 重写，得到正比于 $(\partial h)^2$ 的正则动能项，以及引力子的高阶相互作用。所得拉格朗日量包含引力子相互作用，其耦合常数的能量量纲为负，由此得到了可通过幂次计数论证研究的不可重整化算符。

Early analyses in the 1960s by Feynman [66] and DeWitt [67] showed that the one-loop renormalizability of the theory required the addition of Faddeev-Popov ghosts that, unlike the ordinary ghosts that plague higher-derivative gravity, do not propagate in external legs. Furthermore, in the 1970s, 't Hooft and Veltman [68] extensively studied all the one-loop divergences of Einstein's action and proved that pure gravity without matter, was renormalizable at the one-loop level, while it was not when considering its coupling to matter. The final explicit calculation to confirm the non-renormalization of the theory was performed by Goroff and

Sagnotti [69, 70], who showed that two-loop divergences of pure gravity were unavoidable.

20 世纪 60 年代，费曼 [66] 和德维特 [67] 的早期分析表明，该理论要实现单圈可重整化，需要引入法捷耶夫-波波夫鬼场，这种鬼场与困扰高阶导数引力的普通鬼场不同，它不会在外腿上传播。此外，20 世纪 70 年代，特霍夫特和韦尔特曼 [68] 全面研究了爱因斯坦作用量的所有单圈发散，证明不包含物质的纯引力在单圈水平是可重整化的，而引力与物质耦合后则不具备可重整化性。Goroff 和 Sagnotti[69, 70] 完成了最终的明确计算，证实了该理论的不可重整化性，他们证明纯引力的两圈发散是不可避免的。

From then on, many gravitational theories and approaches have been proposed but, for our purposes, we now comment on some important features of Stelle's gravity (5), whose coupling constants are dimensionless and it results in a satisfactory renormalization of the theory [10,71].

此后，人们提出了诸多引力理论与研究方案，就本文研究而言，我们现在讨论斯泰勒引力 (5) 的几个重要特征: 该理论的耦合常数是无量纲的，最终得到了符合要求的理论重整化结果 [10,71]。

Graviton Propagator

引力子传播子

In order to construct the quantum theory, we begin by computing the graviton propagator of the nonlocal gravity in $D = 4$ dimensions:

为了构建量子理论，我们首先计算 $D = 4$ 维非局部引力的引力子传播子:

$$\mathcal{L} = \frac{1}{2\kappa^2}\left[R + R\gamma_0(\Box)R + R_{\mu\upsilon}\gamma_2(\Box)R^{\mu\upsilon}\right], \tag{145}$$

where we have applied the Gauss-Bonnet theorem to get rid of the Riemann-Riemann tensor term and the form factors $\gamma_0(\Box)$ and $\gamma_2(\Box)$ are defined as (33) in the following way:

此处我们应用高斯-博内定理消去了黎曼-黎曼张量项，形状因子 $\gamma_0(\Box)$ 和 $\gamma_2(\Box)$ 定义如下，即式 (33):

$$\gamma_0(\Box) = \frac{e^{\mathrm{H}_0(\Box)} - 1}{2\Box},\ \gamma_2(\Box) = -\frac{e^{\mathrm{H}_2(\Box)} - 1}{\Box}. \tag{146}$$

Notice the dimensionless couplings $\left[\kappa^{-2}\gamma_{0,2}\right] = 0$.

请注意这些无量纲耦合 $\left[\kappa^{-2}\gamma_{0,2}\right] = 0$ 。

Expanding this nonlocal action to second order in the background-plus-graviton decomposition (98), one obtains [30]

将该非局部作用量按背景 + 引力子分解 (98) 展开到二阶，可得 [30]

$$\mathcal{L}^{(2)} = \mathcal{L}_E^{(2)} + \frac{1}{2}\{\Box h_{\mu\nu}\gamma_2(\Box)\, h^{\mu\nu} - \partial_\mu A^\mu \gamma_2(\Box)\,\partial_\nu A^\nu - F^{\mu\nu}\gamma_2(\Box)\, F_{\mu\nu} +$$

$$\left(\partial_\mu A^\mu - \Box h\right)[\gamma_2(\Box) + 4\gamma_0(\Box)](\partial_\nu A^\nu - \Box h)\},$$

(147)

where we have neglected total derivatives and defined the quantities $A^\mu = \partial_\upsilon h^{\mu\upsilon}$ and $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$, and $\Box = \eta_{\mu\nu}\partial_\mu\partial_\nu$ at this order in the expansion.

此处我们忽略了全导数，并在该展开阶数定义了量 $A^\mu = \partial_\upsilon h^{\mu\upsilon}$ 、$F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ 和 $\Box = \eta_{\mu\nu}\partial_\mu\partial_\nu$ 。

Diffeomorphism invariance constitutes the gauge symmetry of the gravitational theory and it is manifested by the invariance of the Lagrangian (147) under the coordinate transformation [72]

微分同胚不变性是引力理论的规范对称性，体现为拉格朗日量 (147) 在坐标变换下不变 [72]

$$x^\mu \to x^\mu + \zeta^\mu \to h_{\mu\upsilon} \to h_{\mu\upsilon} - 2\partial_{(\mu}\zeta_{\upsilon)} \tag{148}$$

Together with this local symmetry, one introduces a gauge-fixing term at the Lagrangian level given by the harmonic or De Donder gauge [10]

结合该局域对称性，我们在拉格朗日量层面引入规范固定项，采用调和规范即德唐德规范 [10]

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{\xi}A_\mu\omega(\Box)\, A^\mu, \tag{149}$$

where $\omega(\Box)$ is a dimensionless weight function [10, 73] and ξ is the gauge-fixing parameter. Then, the complete Lagrangian is given by

其中 $\omega(\Box)$ 是无量纲权函数 [10, 73] ，ξ 是规范固定参数。因此，完整拉格朗日量为

$$\mathcal{L}^{(2)} + \mathcal{L}_{\mathrm{gf}} = \frac{1}{2}h^{\mu\upsilon}\mathcal{O}_{\mu\upsilon\rho\sigma}h^{\rho\sigma}, \tag{150}$$

where the inverse of $\mathcal{O}_{\mu\upsilon\rho\sigma}$ is the graviton propagator

其中 $\mathcal{O}_{\mu\upsilon\rho\sigma}$ 的逆即为引力子传播子

$$\langle 0\left|\hat{T}\{h_{\mu\nu}(x)\, h_{\rho\sigma}(y)\}\right|0\rangle = i\mathcal{O}^{-1}_{\mu\nu\rho\sigma}. \tag{151}$$

Following the procedure of [30] we find the explicit expression of the graviton propagator in terms of the Barnes-Rivers operators [74,75]

遵循文献 [30] 的步骤，我们得到了引力子传播子用巴恩斯-里弗斯算子表示的显式表达式 [74,75]

$$P^{(0)}_{\mu\upsilon\rho\sigma} = \frac{1}{3}\theta_{\mu\upsilon}\theta_{\rho\sigma}$$

$$P^{(1)}_{\mu\upsilon\rho\sigma} = \theta_{\mu(\rho}\,\omega_{\sigma)\upsilon} + \theta_{\upsilon(\rho}\,\omega_{\sigma)\mu}, \tag{152}$$

$$P^{(2)}_{\mu\upsilon\rho\sigma} = \theta_{\mu(\rho}\,\theta_{\sigma)\upsilon} - \frac{1}{3}\theta_{\mu\upsilon}\theta_{\rho\sigma},$$

$$\bar{P}^{(0)}_{\mu\upsilon\rho\sigma} = \omega_{\mu\upsilon}\omega_{\rho\sigma}$$

where $\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ represents the transversal vector projection operator and $\omega_{\mu\upsilon} = \frac{k_\mu k_\upsilon}{k^2}$ is the longitudinal vector projection operator. Using these operators and their mathematical properties, one derives the graviton propagator of the nonlocal theory (145) in momentum space [30]

其中 $\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ 代表横矢量投影算子，$\omega_{\mu\upsilon} = \frac{k_\mu k_\upsilon}{k^2}$ 是纵矢量投影算子。利用这些算子及其数学性质，可以推导出动量空间中非局部理论 (145) 的引力子传播子 [30]

$$\mathcal{O}^{-1} = -\frac{1}{k^2}\left[\frac{P^{(2)}}{1-k^2\gamma_2\left(k^2\right)} - \frac{P^{(0)}}{2+4k^2\left[3\gamma_0\left(k^2\right)+\gamma_2\left(k^2\right)\right]} + \frac{\xi\left(2P^{(1)}+\bar{P}^{(0)}\right)}{2\omega\left(k^2\right)}\right], \tag{153}$$

and using the convenient definition of the form factors (146) we find

再利用形状因子 (146) 的简便定义，我们得到

$$\mathcal{O}^{-1} = \frac{1}{k^2}\left[\frac{P^{(2)}}{e^{\mathrm{H}_2(k)}} - \frac{P^{(0)}}{2e^{\mathrm{H}_0(k^2)}}\right] - \frac{\xi\left(2P^{(1)}+\bar{P}^{(0)}\right)}{2k^2\omega\left(k^2\right)}, \tag{154}$$

where we have omitted the index structure of the tensors. Assuming the minimal choice $\mathrm{H}_0 = \mathrm{H}_2 \equiv \mathrm{H}$ (34) , the expression (154) becomes [76]

此处我们省略了张量的指标结构。取最小选择 $\mathrm{H}_0 = \mathrm{H}_2 \equiv \mathrm{H}$ (34) 后，表达式 (154) 可化简为 [76]

$$\mathcal{O}^{-1} = \frac{e^{-\mathrm{H}}}{k^2}\left[P^{(2)} - \frac{P^{(0)}}{2}\right] - \frac{\xi\left(2P^{(1)}+\bar{P}^{(0)}\right)}{2k^2\omega\left(k^2\right)}, \tag{155}$$

where the first part of the propagator coincides with the one obtained in Einstein's gravity but with an inverse exponential that improves the UV convergence of the theory. Moreover, note the particular expression of the form factors (146) with the inverse Laplace-Beltrami operator that is canceled by the k^2 factors that go with the form factors in (153), as well as the generality of the propagator, that is valid even for a local theory with polynomials $\mathrm{H}_0(\Box) = p_0(\Box)$ and $\mathrm{H}_2(\Box) = p_2(\Box)$ satisfying $p_0(0) = p_2(0) = 1$.

其中传播子的第一部分与爱因斯坦引力中得到的结果一致, 但多了一个逆指数项改善了理论的紫外收敛性。此外，请注意形状因子 (146) 含逆拉普拉斯-贝尔特拉米算子的特殊形式: 该算子会被 (153) 中与形状因子相乘的 k^2 因子抵消，同时该传播子具有普适性，即使对于满足 $p_0(0) = p_2(0) = 1$. 的含多项式 $\mathrm{H}_0(\Box) = p_0(\Box)$ 和 $\mathrm{H}_2(\Box) = p_2(\Box)$ 的局域理论也成立

Tree-Level Unitarity

树级幺正性

Once introduced the graviton propagator of the nonlocal theory we are in position to prove the unitarity of the theory. In particular, we begin the analysis focusing on tree-level unitarity and finally we briefly recall how to prove the perturbative unitarity of the theory.

在介绍了非局部理论的引力子传播子后，我们已经可以证明该理论的幺正性。具体而言，我们先从分析树级幺正性入手，最后再简要回顾如何证明该理论的微扰幺正性。

We have already mentioned the connection between unitarity and ghost modes at the classical level. However, when dealing with a quantum theory, the presence of ghosts in the particle spectrum is related to the order of poles of the propagator, namely, the theory is unitary if the corresponding propagator has only simple poles in $k^2+m^2=0$, such that the residues of the propagator are positive; a negative residue indicates violation of unitarity as one sees from (13).

我们已经提到了经典层面幺正性与鬼模之间的关联。但在量子理论中，粒子谱中存在鬼与传播子的极点阶数相关，即若对应传播子在 $k^2+m^2=0$ 中仅存在单极点，且传播子的留数为正，则理论是幺正的；正如我们从式 (13) 中看到的，负留数意味着幺正性破缺。

The unitarity test is performed when we couple the graviton to a general conserved stress-energy tensor $T^{\mu\upsilon}$ and check that the scattering amplitudes satisfy the unitarity bound (133). In this approach, the linearized Lagrangian becomes

我们将引力子耦合到一般守恒能动量张量 $T^{\mu\upsilon}$ ，并检验散射振幅满足幺正界 (133)，从而完成幺正性检验。在该方法下，线性化拉格朗日量变为

$$\mathcal{L}_{hT}=\frac{1}{2}h^{\mu\upsilon}\mathcal{O}_{\mu\upsilon\rho\sigma}h^{\rho\sigma}-\sqrt{2}h_{\mu\upsilon}T^{\mu\upsilon}, \tag{156}$$

and one computes the scattering amplitude using the perturbative decomposition (129) and (134) of a initial state i to decay into a final state f [30]

随后人们利用初态 i 衰变为末态 f 的微扰分解 (129) 和 (134) 计算散射振幅 [30]

$$\langle f\left|i\hat{T}\right|i\rangle=(2\pi)^4\delta^{(4)}\left(P_T\right)i\mathcal{M}_{if}=(2\pi)^4\delta^{(4)}\left(P_T\right)i^2T^{\mu\upsilon}i\mathcal{O}^{-1}_{\mu\upsilon\rho\sigma}T^{\rho\sigma}. \tag{157}$$

To find explicit expressions, one decomposes the tensor $T^{\mu\upsilon}$ in terms of the following independent vectors in momentum space:

为得到显式表达式，人们将张量 $T^{\mu\upsilon}$ 按动量空间中如下独立矢量进行分解:

$$k^\mu=\left(k^0,k^i\right),\ \tilde{k}^\mu=\left(-k^0,k^i\right),\ \varepsilon^\mu_i=\left(0,\varepsilon^i\right), \tag{158}$$

for $i = 1, 2$, being ε_i^μ orthogonal to k^μ . The decomposition is done in a completely general way, and one writes

对于 $i = 1, 2$ ，它与 k^μ 正交，即 ε_i^μ 。该分解是完全一般化的，可写为

$$T^{\mu\upsilon} = ak^\mu k^\upsilon + b\tilde{k}^\mu \tilde{k}^\upsilon + c^{ij}\varepsilon_i^{(\mu}\varepsilon_j^{\upsilon)} + dk^{(\mu}\tilde{k}^{\upsilon)} + e^i k^{(\mu}\varepsilon_i^{\upsilon)} + f^i \tilde{k}^{(\mu}\varepsilon_i^{\upsilon)}, \tag{159}$$

and imposing conservation of the stress-energy tensor in momentum space, i.e., $k_\mu T^{\mu\upsilon} = 0$, one finds that $d = b = f^i = 0$. If we want to compute $2\,\mathrm{Im}\,\mathcal{M}_{if}$, we need the imaginary part of

并且对动量空间中的能量动量张量施加守恒条件，即 $k_\mu T^{\mu\upsilon} = 0$ ，可得 $d = b = f^i = 0$。若我们要计算 $2\,\mathrm{Im}\,\mathcal{M}_{if}$ ，则需要得到

$$\mathcal{M}_{if} = -T^{\mu\upsilon} i\mathcal{O}^{-1}_{\mu\upsilon\rho\sigma} T^{\rho\sigma}, \tag{160}$$

where the graviton propagator has been obtained in (154). The gauge-dependent part of the propagator vanishes when contracted with the stress-energy tensors, so that we can neglect this part and rewrite the propagator, including the $+i\varepsilon$ prescription, as

其中引力子传播子已在式 (154) 中给出。传播子的规范依赖部分与能量动量张量缩并后为零，因此我们可以忽略这部分，并将包含 $+i\varepsilon$ 规则的传播子重写为

$$i\mathcal{O}^{-1} = \frac{i}{i\varepsilon - k^2}\left[\frac{P^{(2)}}{e^{\mathrm{H}_2}} - \frac{P^{(0)}}{2e^{\mathrm{H}_0}}\right], \tag{161}$$

so that

由此可得

$$\mathcal{M}_{if} = -T^{\mu\upsilon}\frac{1}{i\varepsilon - k^2}\left[\frac{P^{(2)}}{e^{\mathrm{H}_2}} - \frac{P^{(0)}}{2e^{\mathrm{H}_0}}\right]_{\mu\upsilon\rho\sigma} T^{\rho\sigma}. \tag{162}$$

To calculate the imaginary part of the previous expression, we start focusing on $\frac{1}{i\varepsilon - k^2}$:

为了计算上一个表达式的虚部，我们首先聚焦于 $\frac{1}{i\varepsilon - k^2}$:

$$\mathrm{Im}\left(\frac{1}{i\varepsilon - k^2}\right) = \frac{1}{2i}\left(\frac{1}{i\varepsilon - k^2} - \frac{1}{-i\varepsilon - k^2}\right) = \frac{\varepsilon}{k^4 + \varepsilon^2} \overset{\varepsilon \to 0}{\to} \pi\delta\left(k^2\right). \tag{163}$$

On the other hand, using the formulæ of the Barnes-Rivers operators, one gets

另一方面，利用巴恩斯-里弗斯算子公式可得

$$2\,\mathrm{Im}\,\mathcal{M}_{if} = 2\left[T_{\mu\upsilon}T^{\mu\upsilon} - \frac{T^2}{2}\right]\pi\delta\left(k^2\right), \tag{164}$$

where $T = T^{\mu}_{\mu}$ is the trace of the stress-energy tensor, and we have used the property $\mathrm{H}_0(0) = \mathrm{H}_2(0) = 0$. Finally, using the expression for $T^{\mu\nu}$ with $b = d = f^i = 0$ along with the condition $k_\mu T^{\mu\nu} = 0$, one finds that

其中 $T = T^{\mu}_{\mu}$ 是应力-能量张量的迹，我们利用了性质 $\mathrm{H}_0(0) = \mathrm{H}_2(0) = 0$。最后，将 $T^{\mu\nu}$ 的表达式结合 $b = d = f^i = 0$，并利用条件 $k_\mu T^{\mu\nu} = 0$，可以得到

$$2\,\mathrm{Im}\,\mathcal{M}_{if} = 2\left[\left(c^{ij}\right)^2 - \frac{\left(c^{ii}\right)^2}{2}\right]\pi\delta\left(k^2\right), \tag{165}$$

and since

并且由于

$$c^{ij} = \begin{pmatrix} c^{11} & c^{12} \\ c^{21} & c^{22} \end{pmatrix}, \tag{166}$$

we have that

我们得到

$$\left(c^{ij}\right)^2 - \frac{\left(c^{ii}\right)^2}{2} = \frac{1}{2}\left[\left(c^{11}\right)^2 + \left(c^{22}\right)^2\right] + \left(c^{12}\right)^2 + \left(c^{21}\right)^2. \tag{167}$$

Thus, one concludes that the unitarity bound $2\,\mathrm{Im}\,\mathcal{M}_{if} > 0$ is satisfied and the theory is unitary at the tree-level.

因此，可以得出结论，幺正性界 $2\,\mathrm{Im}\,\mathcal{M}_{if} > 0$ 得到满足，该理论在树图阶是幺正的。

Perturbative Unitarity

微扰幺正性

As we commented on section ”Unitarity”, perturbative unitarity of the nonlocal scalar theory is achieved by means of Efimov analytic continuation together with the verification of the Cutkosky rules. For NLQG, the procedure is analogous but with slightly different Cutkosky rules. The explicit proof of perturbative unitarity is tedious [77, 78] but, intuitively, one may reason that since the nonlocal theory only affects the Einstein’s graviton propagator with the inclusion of the form factors, which do not produce extra poles, no unitarity violation is induced.

正如我们在“幺正性”一节中的讨论，非局域标量理论的微扰幺正性是通过叶菲莫夫解析延拓，并结合 Cutkosky 规则的验证实现的。对于非局域量子引力 (NLQG)，这一过程是类似的，仅 Cutkosky 规则略有不同。微扰幺正性的显式证明十分繁琐 [77, 78]，但从直观上可以推断: 由于非局域理论仅通过引入形状因子修改爱因斯坦引力子传播子，而形状因子不会产生额外极点，因此不会引发幺正性破坏。

Tree-Level Scattering Amplitudes

树级散射振幅

Once sketched the proof of the unitarity of the nonlocal theory, in this part we recall the calculation of some graviton scattering amplitudes at tree-level for both Stelle's gravity and NLQG using the convenient transverse traceless gauge for the graviton field. Furthermore, we show an important result that relates the scattering amplitudes of these theories at tree-level with the ones computed using Einstein's gravity [79].

在概述了非局部理论的幺正性证明后，本部分我们将使用对引力子场更简便的横向无迹规范，回顾斯泰勒引力与非局部量子引力中几种树级引力子散射振幅的计算。此外，我们还会展示一项重要结果：这些理论的树级散射振幅与爱因斯坦引力中计算得到的结果相关 [79]。

First, in order to calculate scattering amplitudes conveniently, we work in the transverse traceless gauge for the graviton field, so that any on-shell graviton in the external legs of the scattering process satisfies the gauge-fixing conditions

首先，为了方便计算散射振幅，我们对引力子场采用横向无迹规范，使得散射过程外线的所有在壳引力子都满足规范固定条件

$$\partial^\mu h_{\mu v} = 0 \; h = h_\mu^\mu = 0. \tag{168}$$

We now define the dimensionless graviton field $[h] = 0$ as

现在我们定义无量纲引力子场 $[h] = 0$ 为

$$g_{\mu v} = \eta_{\mu v} + h_{\mu v}. \tag{169}$$

In this gauge, we may decompose the polarization tensor $\varepsilon_{\mu\nu}$ of the spin-2 graviton as a combination of the spin-1 photon polarization vectors ε_μ as [41]

在该规范下，我们可以将自旋 2 引力子的极化张量 $\varepsilon_{\mu\nu}$ 分解为自旋 1 光子极化矢量 ε_μ 的组合，形式如下 [41]

$$\varepsilon_{\mu v}(p, \pm 2) = \varepsilon_\mu(p, \pm 1)\, \varepsilon_v(p, \pm 1), \tag{170}$$

with

其中

$$\varepsilon_\mu(p, \lambda)\, p^\mu = 0 \; \varepsilon_\mu(\lambda)\, \varepsilon^\mu(\lambda) = 0. \tag{171}$$

We define the helicity amplitudes $F_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ for a 4 graviton scattering $1 + 2 \to 3 + 4$ by the relation with the already introduced scattering amplitude $\mathcal{M}$ via

我们通过与已引入的散射振幅 $\mathcal{M}$ 的关系，定义 4 引力子散射 $1+2 \to 3+4$ 的螺旋度振幅 $F_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ 为

$$F_{\lambda_3,\lambda_4;\lambda_1,\lambda_2} = \frac{i}{\mathcal{N}} \langle \lambda_3, \lambda_4 \left| \mathcal{M} \right| \lambda_1, \lambda_2 \rangle , \tag{172}$$

where $\mathcal{N}$ is a normalization factor.

其中 $\mathcal{N}$ 为归一化因子。

The introduction of these helicity amplitudes allows one to obtain four handy properties for this particular 2-2 scattering process, depicted in Fig. 5.

引入这些螺旋度振幅后，可以得到该特定 2-2 散射过程的四个简便性质，如图 5 所示。

Imposing the individual discrete symmetries C, P , and T , together with Bose symmetry because of the bosonic character of the gravitons, one finds the following four properties of helicity amplitudes [80]:

对引力子分别施加离散对称性 C, P 和 T ，再结合引力子作为玻色子满足的玻色对称性，我们可以得到螺旋度振幅的如下四个性质 [80]:

- P symmetry:

- P 对称性:

$$F_{\lambda_3,\lambda_4;\lambda_1,\lambda_2} = (-1)^{\lambda-\mu} F_{-\lambda_3,-\lambda_4;-\lambda_1,-\lambda_2}. \tag{173}$$

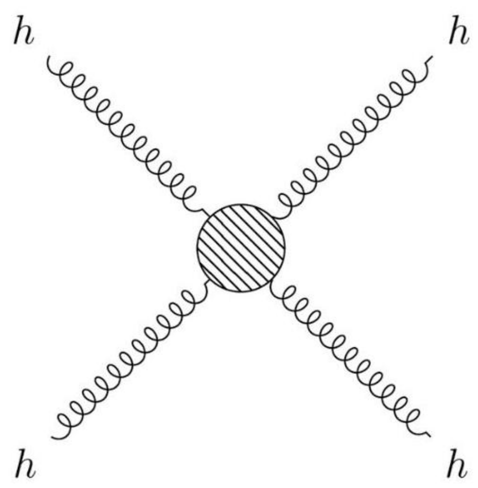
- T symmetry in the process $a+b \to a+b$:

- 过程 $a+b \to a+b$ 中的 T 对称性:

$$F_{\lambda_3,\lambda_4;\lambda_1,\lambda_2} = (-1)^{\lambda-\mu} F_{\lambda_1,\lambda_2;\lambda_3,\lambda_4}. \tag{174}$$

Fig. 5 2-2 graviton scattering process

图 5 2-2 引力子散射过程



- C symmetry in the process $a+\bar{a}\to b+\bar{b}$:

> - 过程 $a+\bar{a}\to b+\bar{b}$ 中的 C 对称性:

$$F_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}=(-1)^{\lambda-4}F_{\lambda_3,\lambda_4;\lambda_2,\lambda_1}. \tag{175}$$

- Bose symmetry in the process $a+a\to b+c$:

> - 过程 $a+a\to b+c$ 中的玻色对称性:

$$F_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}=(-1)^{\lambda-\mu}F_{\lambda_4,\lambda_3;\lambda_2,\lambda_1}, \tag{176}$$

where $\lambda=\lambda_1-\lambda_2$ and $\mu=\lambda_3-\lambda_4$. Taking into account these properties, we find that out of 16 initial independent components, we are only left with 4:

> 其中 $\lambda=\lambda_1-\lambda_2$ 和 $\mu=\lambda_3-\lambda_4$ 。考虑这些性质后我们发现，初始的 16 个独立分量中，仅剩下 4 个:

$$\mathcal{A}(++;++)\equiv F_{2,2;2,2},\ \mathcal{A}(+-;+-)\equiv F_{2,-2;2,-2}, \tag{177}$$

$$\mathcal{A}(++;+-)\equiv F_{2,2;2,-2},\ \mathcal{A}(++;--)\equiv F_{2,2;-2,-2}.$$

Helicity Amplitudes in Local Gravity

> ## 局域引力中的螺旋度振幅

For the action (5), using the perturbative decomposition (169) one now computes the 3- and 4-point functions of the theory, expanding up to third and fourth order in the graviton $h_{\mu\nu}$. Besides, since we aim to compute the 2-2 scattering amplitudes, for the sake of simplicity we may assume two on-shell gravitons in the following derivations so that the gauge conditions (168) may be imposed onto two out of three of the gravitons present in the 3-point functions and on the four gravitons present in the 4-point function. In particular, imposing on-shell conditions in the external gravitons determines that the linearized Ricci scalar and the determinant of the metric vanish,

对于作用量 (5)，我们利用微扰分解 (169) 计算该理论的 3 点函数和 4 点函数，展开到引力子 $h_{\mu\nu}$ 的三阶和四阶。此外，由于我们的目标是计算 2-2 散射振幅，为简化计算，我们可在后续推导中假设存在两个在壳引力子，从而可将规范条件 (168) 应用于 3 点函数中三个引力子的其中两个，以及 4 点函数中全部四个引力子。具体而言，对外部引力子施加在壳条件可以确定线性化里奇标量和度规的行列式均为零，

$$R^{(1)} = \eta^{\mu\nu} R^{(1)}_{\mu\nu} = \partial_\alpha \partial_\beta h^{\alpha\beta} - \Box h = 0, \tag{178}$$

$$\sqrt{-g}^{(1)} = \sqrt{-h}^{(1)} = 0. \tag{179}$$

Moreover, the linearized equations of motion imply that the Ricci tensor at linear order vanishes as well,

此外，线性化运动方程表明一阶线性化里奇张量也为零，

$$\left(\frac{\delta S}{\delta g_{\mu\nu}}\right)^{(1)} = 0 \Rightarrow R^{(1)}_{\mu\nu} = \frac{1}{2}\left(-\partial_\mu \partial_\nu h + \partial_\mu \partial^\alpha h_{\alpha\nu} + \partial_\nu \partial^\alpha h_{\alpha\mu} - \Box h_{\mu\nu}\right) = 0.$$

(180)

Furthermore, using the Gauss-Bonnet theorem we may rewrite Stelle's action (5) as

进一步，利用高斯-博内定理，我们可以将施泰勒作用量 (5) 重写为

$$S_{\text{Stelle}} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(R + \gamma_0 R^2 + \gamma_2 R_{\mu\nu} R^{\mu\nu}\right) = S_{\text{E}} + \gamma_0 S_0 + \gamma_2 S_2, \tag{181}$$

where $\gamma_0 = \alpha_1 - \alpha_3$ and $\gamma_2 = \alpha_2 + 4\alpha_3$.

其中 $\gamma_0 = \alpha_1 - \alpha_3$ 和 $\gamma_2 = \alpha_2 + 4\alpha_3$ 。

Equations of Motion

运动方程

For the 3-point function, one needs to expand (181) at second order and use the vanishing values for the Ricci Scalar, Ricci tensor and the trace of the graviton at the linear order. From these expressions, we may find the 3-point functions treating the off-shell graviton $\widetilde{h}_{\mu\nu}$ as an independent field unconstrained by the gauge conditions. Using this prescription, one finds that each term of the Lagrangian (181) gives rise to the following equations of motion at second order [79]:

为了计算三点函数，我们需要将式 (181) 展开到二阶，并利用里奇标量、里奇张量以及引力子迹在线性阶为零的条件。利用这些表达式，我们可以将离壳引力子 $\tilde{h}_{\mu\nu}$ 视为不受规范条件约束的独立场，从而得到三点函数。根据这套处理方法，可以得到拉格朗日量 (181) 的每一项在二阶给出如下运动方程 [79]:

$$\left(\frac{\delta S_{\mathrm{E}}}{\delta g_{\mu\upsilon}}\right)^{(2)}=\frac{1}{2\kappa^2}\left(\frac{1}{2}g^{\mu\upsilon}R-R^{\mu\upsilon}\right)^{(2)}=\frac{1}{2\kappa^2}\left[\frac{1}{2}\eta^{\mu\upsilon}R^{(2)}-(R^{\mu\upsilon})^{(2)}\right].$$

$$\left(\frac{\delta S_0}{\delta g_{\mu\nu}}\right)^{(2)}=\frac{1}{2\kappa^2}\gamma_0\left[2\sqrt{-g}\left(g^{\mu\upsilon}\nabla^2-\nabla^{\mu}\nabla^{\upsilon}+\frac{1}{4}g^{\mu\upsilon}R-R^{\mu\upsilon}\right)R\right]^{(2)}$$

$$=\frac{\gamma_0}{\kappa^2}\left(\eta^{\mu\upsilon}\Box-\partial^{\mu}\partial^{\upsilon}\right)R^{(2)}.$$

$$\left(\frac{\delta S_2}{\delta g_{\mu\nu}}\right)^{(2)}=\frac{\gamma_2}{2\kappa^2}\left[\sqrt{-g}\left(\frac{1}{2}g^{\mu\upsilon}R^{\alpha\beta}R_{\alpha\beta}-2R^{\mu\alpha}R^{\upsilon}_{\alpha}\right)\right]^{(2)}+$$

$$+\frac{\gamma_2}{2\kappa^2}\left[\sqrt{-g}\left(-g^{\mu\alpha}g^{\nu\beta}\nabla^2-g^{\mu\nu}\nabla^{\alpha}\nabla^{\beta}\right)R_{\alpha\beta}\right]^{(2)}+$$

$$+\frac{\gamma_2}{2\kappa^2}\left[\sqrt{-g}\left(g^{\mu\alpha}\nabla^{\beta}\nabla^{\nu}+g^{\nu\alpha}\nabla^{\beta}\nabla^{\mu}\right)R_{\alpha\beta}\right]^{(2)}+$$

$$=\frac{\gamma_2}{2\kappa^2}\left(-\eta^{\mu\alpha}\eta^{\nu\beta}\Box-\eta^{\mu\upsilon}\partial^{\alpha}\partial^{\beta}+\eta^{\mu\alpha}\partial^{\beta}\partial^{\upsilon}+\eta^{\upsilon\beta}\partial^{\alpha}\partial^{\mu}\right)R^{(2)}_{\alpha\beta},$$

where the Ricci tensor and the Ricci scalar at second order in the perturbation are given by

其中微扰二阶下的里奇张量和里奇标量由下式给出

$$R^{(2)}_{\mu\upsilon}=\frac{1}{2}\partial_{\mu}h^{\alpha\beta}\partial_{\upsilon}h_{\alpha\beta}+h^{\alpha\beta}\left(\partial_{\beta}\partial_{\alpha}h_{\mu\upsilon}+\partial_{\mu}\partial_{\upsilon}h_{\alpha\beta}-\partial_{\beta}\partial_{\alpha}h_{\upsilon\alpha}-\partial_{\beta}\partial_{\upsilon}h_{\mu\alpha}\right)$$

$$+\partial^{\beta}h^{\alpha}_{\mu}\left(\partial_{\beta}h_{\upsilon\alpha}-\partial_{\alpha}h_{\upsilon\beta}\right),$$

$$R^{(2)}=-\partial_{\beta}h_{\alpha\sigma}\partial^{\sigma}h^{\alpha\beta}+\frac{3}{2}\partial_{\sigma}h_{\alpha\beta}\partial^{\sigma}h^{\alpha\beta}.$$

(182)

Similarly, for the on-shell graviton interactions, we only need to reproduce the previous analysis but now expanding the equations of motion to fourth order [79].

类似地，对于在壳引力子相互作用，我们只需要重复上述分析，只是此处需要将运动方程展开到四阶 [79]。

4-Graviton Scattering Amplitudes at Tree-Level

树级 4 引力子散射振幅

In this subsection, we recall the scattering amplitudes for the four available channels shown in Fig. 6 of the 2-2 graviton scattering, and to do so we make use of the Mandelstam variables for the process $p_1 + p_2 \to p_3 + p_4$, defined as

在本小节中，我们回顾图 6 所示 2-2 引力子散射四个可用道的散射振幅，为此我们对过程 $p_1 + p_2 \to p_3 + p_4$ 使用定义如下的曼德尔施塔姆变量

$$s = -(p_1 + p_2)^2 = 4E^2$$

$$t = -(p_1 - p_3)^2 = -2E^2\,(1 - \cos\theta)\,, \tag{183}$$

$$u = -(p_1 - p_4)^2 = -2E^2\,(1 + \cos\theta)\,,$$

where E and θ are the energy and the scattering angle in the centre-of-mass reference frame.

其中 E 和 θ 分别是质心系中的能量和散射角

With all the necessary ingredients, one may proceed to calculate the four independent helicity amplitudes (177). Adding the four contributions depicted in Fig. 6, one finds [79]

准备好所有必要要素后，就可以计算四个独立螺旋度振幅 (177)。将图 6 所示的四个贡献相加，可得 [79]

$$\mathcal{A}\,(++,++) = \frac{i}{\kappa^2}\frac{E^2}{\sin^2\theta} = \mathcal{A}_{\mathrm{E}}\,(++,++)\,,$$

$$\mathcal{A}\,(++,+-) = 0 = \mathcal{A}_{\mathrm{E}}\,(++,++)\,, \tag{184}$$

$$\mathcal{A}\,(+-,+-) = \frac{1}{8}\frac{i}{\kappa^2}\frac{E^2(1+\cos\theta)^4}{\sin^2\theta} = \mathcal{A}_{\mathrm{E}}\,(+-,+-)\,,$$

$$\mathcal{A}\,(++,--) = 0 = \mathcal{A}_{\mathrm{E}}\,(++,++)\,.$$

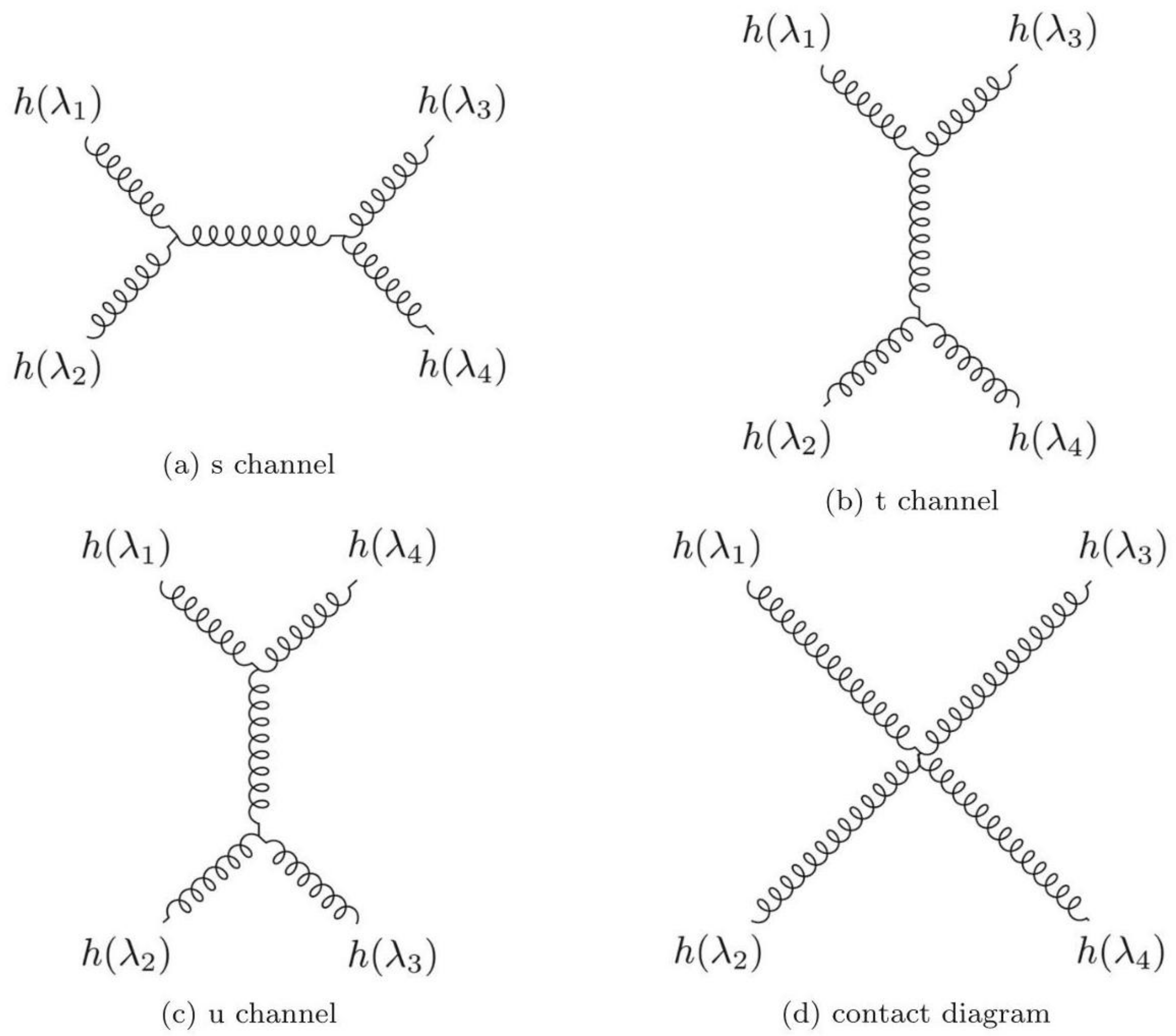


Fig. 6 The four contributions to the tree-level 2-2 graviton scattering. (a) s channel. (b) t channel. (c) u channel. (d) contact diagram

图 6 树级 2-2 引力子散射的四个贡献。(a) s 道；(b) t 道；(c) u 道；(d) 接触图

The most impressive result is that, at the tree-level, the scattering amplitudes of Stelle's gravity coincide with the ones computed in Einstein's theory [81].

最引人注目的结果是，在树级，施泰勒引力的散射振幅与爱因斯坦理论中计算得到的振幅一致 [81]

Helicity Amplitudes in Nonlocal Gravity

非局部引力中的螺旋度振幅

For the action (145), we may proceed analogously to what done in Stelle's gravity. In fact, since for the 4-graviton scattering amplitude we only need to compute the action at second and at fourth order in the perturbation $h_{\mu\nu}$ and since nonlocality modifies the terms proportional to $R_{\mu\nu}^2$ and R^2 and we have shown that these are at least quadratic in the perturbation, at the end of the day, the results obtained for the nonlocal case are the same than the higher-derivative theory. The nonlocal form factors will be mere spectators in the development because, at the order of the expansion we are interested in, they do not play any role so that the expressions of the scattering amplitudes remain intact and one recovers the values (184) [79]. This result is a key to ensure causality.

对于作用量 (145)，我们可以按照斯蒂尔引力中的推导方法类似进行。事实上，对于 4 引力子散射振幅，我们仅需要计算作用量在微扰 $h_{\mu\nu}$ 下的二阶和四阶项，而非局部性修正了正比于 $R_{\mu\nu}^2$ 和 R^2 的项，并且我们已经证明这些项至少是微扰的二次项，最终非局部情形得到的结果和高阶导数理论完全一致。非局部形状因子在推导过程中仅作为旁观者，因为在我们所关注的展开阶数下，它们不发挥任何作用，因此散射振幅的表达式保持不变，我们可以重新得到结果 (184) [79]。这一结果是保证因果性的关键。

Causality

因果性

Often, nonlocality has been associated with acausality, a property that rules out any theory by cause-effect violation; indeed, in the construction of a QFT via the second quantization, one requires causality in the sense that disconnected spacetime events cannot influence each other. However, this connection between acausality and nonlocality is not necessarily true and it can be shown that it does not hold in NLQG by reviewing Shapiro's time delay test [1].

非定域性常被和非因果性关联在一起，后者因违反因果关系会直接否定任何理论；事实上，在用二次量子化构造量子场论时，我们要求理论满足因果性，即不连通的时空事件无法相互影响。但非因果性和非定域性的这种关联并不必然成立，我们可以通过回顾夏皮洛时间延迟检验 [1] 证明，该关联在非定域量子引力 (NLQG) 中不成立。

Shapiro's time delay is considered to be one of the four classic tests of GR, and it states that light propagating near a massive object will suffer a time delay with respect to the measured time while propagating in flat spacetime; this measurable delay is what is called Shapiro’s time delay Δt and it can be used to prove causality by showing that $\Delta t > 0$. A negative value of this measurable quantity would imply a violation of macro-causality.

夏皮洛时间延迟被认为是广义相对论的四大经典检验之一，该效应指出，相较于在平直时空中传播，靠近大质量天体传播的光会产生时间延迟；这种可测量的延迟就是夏皮洛时间延迟 Δt ，它可以通过证明 $\Delta t > 0$ 来验证因果性。该可测量量若为负值，就意味着宏观因果性被破坏。

To compute Shapiro's time delay one only needs to know the tree-level scattering amplitudes [82, 83] and, as we have seen, for both Stelle's gravity and nonlocal quantum gravity one gets the same values of Einstein's theory. Since the causality of GR has been proven both theoretically and experimentally [84], we conclude that NLQG does not induce macroscopic acausality. Microscopic but observationally harmless violations of causality are still possible [85, 86] .

计算夏皮洛时间延迟只需要知道树级散射振幅 [82, 83]，正如我们所见，斯泰勒引力和非定域量子引力得到的结果都和爱因斯坦广义相对论一致。既然广义相对论的因果性已经得到理论和实验的双重证明 [84]，我们可以得出结论: 非定域量子引力不会引发宏观非因果性。微观层面的、不影响观测的因果性破坏仍然是可能存在的 [85, 86] 。

Renormalization

重正化

In this section, we study the renormalizability of NLQG by reviewing the superficial degree of divergence, as well as showing how multiplicative renormalization works in this theory. We also introduce the form of the so-called killer potential at least cubic in $\mathcal{R}$ that is required for the finiteness of the theory in $D = 4$ dimensions. Let us start by commenting on the choice for the form factor, more precisely, $\mathrm{H}(\Box)$. Although, the Wataghin and Krasnikov form factors are very insightful to show the behaviour of the form factors in the UV limit, in practice, one makes use of the asymptotically polynomial form factors, since their growth in the UV is well-defined by a polynomial $p(\Box)$ of degree $n_{\deg}$, so that the calculations become more straightforward.

本节中，我们通过综述表面发散度，说明乘性重正化在该理论中的工作方式，研究非局域量子引力 (NLQG) 的可重正化性。我们还介绍了为使理论在 $D = 4$ 维下有限所需的，对 $\mathcal{R}$ 而言至少为三次的所谓杀伤势形式。我们首先对形状因子的选择展开说明，更确切地说，是 $\mathrm{H}(\Box)$ 。尽管瓦塔金和克拉斯尼科夫形状因子对于展示形状因子在紫外极限下的行为颇具启发性，但实际应用中人们通常采用渐近多项式形状因子，因为它们在紫外的增长由次数为 $n_{\deg}$ 的多项式 $p(\Box)$ 明确定义，因此计算会更简便。

We focus on the minimal theory

我们聚焦于极小理论

$$\mathcal{L} = \frac{1}{2\kappa^2}\left[R + G_{\mu\nu}\frac{e^{\mathrm{H}(\Box)} - 1}{\Box}R^{\mu\nu} + \mathcal{V}(\mathcal{R})\right]. \tag{185}$$

It was shown [76] that one may split (185) into its local and nonlocal part and that the nonlocal part of the Lagrangian does not contribute to the UV divergences of the theory but only to the finite part of its quantum effective action. Therefore, in order to analyse the infinities of the theory, one needs only to consider the local part, given by the UV limit of the previous Lagrangian, with $e^{\mathrm{H}(\Box)} \sim p(\Box)$. Because of this argument, to study the power-counting renormalizability of the theory, one focuses on the local action

已有研究 [76] 表明，可将式 (185) 拆分为局域部分和非局域部分，拉格朗日的非局域部分不会导致理论出现紫外发散，仅对量子有效作用量的有限部分有贡献。因此，分析该理论的无穷大时，只需考虑由前述拉格朗日在紫外极限下得到的局域部分即可，这里涉及 $e^{\mathrm{H}(\Box)} \sim p(\Box)$ 。基于这一论证，研究该理论的幂次计数可重正化性时，我们将目光放在局域作用量上

$$S_{\text{local}} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R + \sum_{n=0}^{n_{\deg}-1} R a_n \Box^n R + \sum_{n=0}^{n_{\deg}-1} R_{\mu\nu} b_n \Box^n R^{\mu\nu}\right],$$

(186)

where for the time being we have ignored $\mathcal{V}(\mathcal{R})$.

此时我们暂时忽略了 $\mathcal{V}(\mathcal{R})$ 。

We can choose the polynomial of the action (186) to be a monomial of the highest order so that the UV behaviour is characterized by the dependence

我们可以将作用量 (186) 的多项式选为最高阶的单项式，这样紫外行为就由依赖关系刻画

$$\mathcal{R}\Box^{n_{\text{deg}}-1}\mathcal{R} \, .$$

To perform an analysis about the superficial degree of divergence of the theory, we first need to account for the non-Abelian character of the gravitational theory. In this sense, in order to quantize via the path integral one needs to introduce three Faddeev-Popov ghost fields [73], which appear in the loops of the theory.

为了分析该理论的表面发散度，我们首先需要考虑引力理论的非阿贝尔性质。从这一点来说，若要通过路径积分进行量子化，我们需要引入三个法捷耶夫-波波夫鬼场 [73]，它们会出现在理论的圈图中。

The superficial degree of divergence of the theory (186) is given by [76]

理论 (186) 的表面发散度由下式给出 [76]

$$\omega(\mathcal{F}) = 4 - 2\left(n_{\text{deg}} - 1\right)(L - 1)\,, \tag{187}$$

and from this expression we notice that assuming

从该表达式我们可以注意到，若假设

$$n_{\text{deg}} > 3\,, \tag{188}$$

only one-loop divergences survive, obtaining a super-renormalizable theory.

仅单圈发散留存，由此得到超可重正化理论。

Stelle’s Gravity

施特勒引力

The previous analysis can be generalized to other local theories such as Stelle’s gravity in $D = 4$ dimensions. For this theory, we have that $n_{\text{deg}} = 1$, so that at one loop

前述分析可以推广到其他局域理论，例如 $D=4$ 维的施特勒引力。对于该理论，我们得到 $n_{\text{deg}}=1$ ，因此单圈层面

$$\omega(\mathcal{F})=4 \tag{189}$$

However, all the apparent divergences of the theory can be reabsorbed by the quadratic operators appearing in the action, resulting in a successful renormalizable theory [10]. This example shows how the power-counting argument is insufficient to fully determine the renormalization properties of a QFT.

然而，该理论所有表观发散都可以被作用量中存在的二次算符重新吸收，最终得到一个成功的可重整化理论 [10]。这个例子说明幂次计数论证不足以完全确定量子场论的重整化性质。

Finiteness

有限性

One-loop divergences may be reabsorbed by the quadratic terms in the curvature in the Lagrangian, resulting in a renormalizable theory. Furthermore, it turns out that we can even define a finite quantum field theory by introducing a suitable potential, in which all the β -functions vanish. More precisely, one requires to introduce the killer operators [76]

单圈发散可以被拉格朗日量中的曲率二次项重新吸收，得到可重整化的理论。此外，我们可以通过引入合适的势来定义一个有限量子场论，其中所有 β 函数都等于零。更准确地说，我们需要引入湮灭算符 [76]

$$\mathcal{V}(\mathcal{R})=s_R^{(1)}R_{\mu\nu}R^{\mu\nu}\Box^{n_{\text{deg}}-3}R_{\rho\sigma}R^{\rho\sigma}+s_R^{(2)}R^2\Box^{n_{\text{deg}}-3}R^2. \tag{190}$$

The non-running coupling constants $s_R^{(1)}$ and $s_R^{(2)}$ only contribute linearly to the β -functions, so that one may always make a suitable choice to achieve finiteness, i.e., the absence of divergences at any loop order.

非跑动耦合常数 $s_R^{(1)}$ 和 $s_R^{(2)}$ 仅对 β 函数做线性贡献，因此我们总可以通过合适的选择实现有限性，即在任意圈阶都不存在发散。

Multiplicative Renormalization

乘性重整化

Let us now sketch the multiplicative regularization using dimensional regularization prescription [59], in which one performs the renormalization procedure in a general dimension $d=4-\varepsilon$ and, subsequently, takes the limit $\varepsilon\to 0$ so that the divergent part of the theory is encoded in terms proportional to $1/\varepsilon$. Afterwards, one reabsorbs these divergences with the counterterms induced by the multiplicative prescription, in which one redefines all the bare coupling constants with a perturbative expansion in terms of the renormalized coupling constants via

现在我们利用维数正则化方案概述乘性正则化 [59]: 该方案在一般维度 $d = 4 - \varepsilon$ 下执行重整化过程，随后取极限 $\varepsilon \to 0$，使理论的发散部分以正比于 $1/\varepsilon$ 的项编码，之后再通过乘性方案引入的抵消项重新吸收这些发散；在该方案中，我们通过如下方式将所有裸耦合常数用重整化耦合常数进行微扰展开重新定义:

$$\lambda_B = Z_\lambda \lambda_R \tag{191}$$

In this sense, after applying multiplicative renormalization one ends up with the renormalized Lagrangian

就此而言，应用乘性重整化后，我们最终得到重整化拉格朗日量:

$$\mathcal{L}^R = \mathcal{L} + \mathcal{L}_{\text{ct}}$$

$$= \mathcal{L} + (Z_{\kappa^{-2}} - 1)\frac{R}{2\kappa^2} + \left(Z_{a_0} - 1\right) a_n R^2 + \left(Z_{b_0} - 1\right) b_n R_{\mu\nu} R^{\mu\nu}, \tag{192}$$

where in dimensional regularization one can write the Lagrangian of the 1-loop counterterms as a function of the β -functions of the coupling constants:

在维数正则化中，我们可以将单圈抵消项的拉格朗日量写为耦合常数 β 函数的函数:

$$\mathcal{L}_{\text{ct}} = \frac{1}{\varepsilon}\left[\frac{1}{2}\beta_{\kappa^{-2}} R + \beta_{a_0} R^2 + \beta_{b_0} R_{\mu\nu} R^{\mu\nu}\right]. \tag{193}$$

Since the local part of $\mathcal{L}$ contains the divergences of the theory in a set of terms we collectively denote as $\mathcal{L}_\infty$, they may be reabsorbed by setting

由于 $\mathcal{L}$ 的局域部分将理论的发散包含在一组我们统一记为 $\mathcal{L}_\infty$ 的项中，因此可以通过下式重新吸收这些发散:

$$\mathcal{L}_{\text{ct}} = -\mathcal{L}_\infty \tag{194}$$

resulting in a renormalizable theory.

最终得到一个可重整化理论。

Comparing (192) and (193), one writes for the coupling constant $\alpha_i = \left\{\kappa^{-2}, a_0, b_0\right\}$

对比 (192) 和 (193)，我们可以对耦合常数 $\alpha_i = \left\{\kappa^{-2}, a_0, b_0\right\}$ 写出:

$$\left(Z_{\alpha_i} - 1\right)\alpha_i = \frac{1}{\varepsilon}\beta_{\alpha_i} \;\Rightarrow\; Z_{\alpha_i} = 1 + \frac{1}{\varepsilon}\beta_{\alpha_i}\frac{1}{\alpha_i}. \tag{195}$$

However, since the vertices $\mathcal{R}^2$ do not give rise to divergences at one loop, one concludes that their β -functions are constant. In this way, we may solve analytically the renormalization-group equations to find

但由于顶点 $\mathcal{R}^2$ 在单圈阶不会产生发散，因此可以得出结论: 它们的 β 函数为常数。由此我们可以解析求解重整化群方程，得到:

$$\beta_{\alpha_i} = \frac{1}{\mu}\frac{d\alpha_i}{d\mu} \Rightarrow \alpha_i(\mu) = \alpha_i(\mu_0) + \beta_{\alpha_i}\log\frac{\mu}{\mu_0}, \tag{196}$$

where μ is the energy scale. From (196), one notices that all the three couplings exhibit the same running in energies.

其中 μ 为能量标度。从 (196) 中可以看出，三个耦合常数随能量的跑动行为完全一致。

Asymptotic Freedom

渐近自由

In (196), one notices the small growth of the coupling constants α_i with the energy scale μ . This hints at the property of asymptotic freedom: since the kinetic term of the theory grows higher in the UV limit than the couplings do, the interactions of NLQG become negligible in the UV. We can show mathematically such property for the minimal theory [29]

在文献 (196) 中，我们可以看到耦合常数 α_i 随能标 μ 的增长十分缓慢。这暗示了渐近自由的性质: 由于该理论的动能项在紫外极限下增长快于耦合，非局部量子引力 (NLQG) 的相互作用在紫外区可忽略不计。我们可以对极小理论从数学上证明这一性质 [29]

$$\begin{aligned} S &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R + G_{\mu\nu}\frac{e^{\mathrm{H}(\Box)}-1}{\Box}R^{\mu\nu}\right] \\ &= \int d^4x\sqrt{-g}\left[\omega R + \sum_{n=0}^{\infty}\left(a_n R\Box^n R + b_n R_{\mu\nu}\Box^n R^{\mu\nu}\right)\right], \end{aligned} \tag{197}$$

where only $\omega(\mu)$, $a_0(\mu)$, and $b_0(\mu)$ have an energy dependence (196), and where $e^{\mathrm{H}(\Box)}$ is assumed to be asymptotically a monomial for simplicity. Expanding this action as a function of the dimensionless graviton (169), one obtains [87]

其中只有 $\omega(\mu), a_0(\mu)$ 和 $b_0(\mu)$ 存在能量依赖关系 (196)，且为简化起见假设 $e^{\mathrm{H}(\Box)}$ 渐近为单项式。将该作用量按无量纲引力子 (169) 展开后，可得 [87]

$$\begin{aligned} S = \int d^4x\{&\omega\left[h\Box h + h^2\Box h + \mathcal{O}(h^4)\right] \\ &+ \sum_{n=0}^{\infty} a_n\left[h\Box^{n+2}h + h^2\Box^{n+2}h + \mathcal{O}(h^4)\right] \\ &+ \sum_{n=0}^{\infty} b_n\left[h\Box^{n+2}h + h^2\Box^{n+2}h + \mathcal{O}(h^4)\right]\Big\}, \end{aligned} \tag{198}$$

where we have omitted the tensorial indices. A suitable rescaling of the graviton field given by

此处我们省略了张量指标。对引力子场做合适的重标度如下:

$$h_{\mu\nu} \to \frac{1}{\sqrt{b_0(\mu)}} h_{\mu\nu} \equiv f(t)\, h_{\mu\nu},\ f^2 = \frac{f_0^2}{1 + f_0^2 \beta_b \log \frac{\mu}{\mu_0}}, \tag{199}$$

allows us to infer the properties of the theory in the UV. Under this rescaling, $[h] = 1$ and the action (198) becomes

这让我们可以推导出该理论在紫外区的性质。在此重标度下，$[h] = 1$ 和作用量 (198) 变为

$$\begin{aligned} S = \int d^4x \{ & \omega \left[f^2 h \Box h + f^3 h^2 \Box h + \mathcal{O}\left(f^4 h^4\right) \right] \\ & + \sum_{n=0}^{\infty} a_n \left[f^2 h \Box^{n+2} h + f^3 h^2 \Box^{n+2} h + \mathcal{O}\left(f^4 h^4\right) \right] \\ & + \sum_{n=0}^{\infty} b_n \left[f^2 h \Box^{n+2} h + f^3 h^2 \Box^{n+2} h + \mathcal{O}\left(f^4 h^4\right) \right] \Big\}. \end{aligned} \tag{200}$$

From this expression, one notes that since $f \to 0$ when $\mu \to \infty$, the action at leading order involves only two gravitons, so that all interactions are suppressed in the UV limit. Furthermore, the rescaling (199) ensures the validity of perturbation theory in the high-energy regime.

从该表达式可以看出，由于当 $\mu \to \infty$ 时的 $f \to 0$ ，领头阶作用量仅包含两个引力子，因此所有相互作用在紫外极限下都被抑制。此外，重标度 (199) 保证了高能区微扰论的有效性。

Conclusions and Prospects

结论与展望

In this chapter, we have reviewed Stelle's gravity, a theory that successfully removes the quantum divergences of GR by the introduction of higher-derivative terms in the action. However, these terms lead to instabilities both at the classical and quantum level. Therefore, in order to solve this breaking of unitarity, we have motivated the introduction of nonlocal gravity. Preserving unitarity drives us to consider exponential and asymptotically polynomial nonlocal operators, whose construction is based on: (i) the recovery of GR at low energies, (ii) the desired behaviour of the propagator of the theory in the UV so that renormalizability and even finiteness of the theory are achieved, and finally (iii) the classical singularity problem of Einstein's theory may be overcome. Although the introduction of nonlocality in the theory may be deemed as problematic because it obscures the solution of the problem of initial conditions, the diffusion method provides a rigorous prescription to clarify these apparent problems.

在本章中，我们回顾了斯特勒引力——该理论通过在作用量中引入高阶导数项，成功消除了广义相对论的量子发散。不过，这些高阶项会在经典和量子层面都引发不稳定性。因此，为了解决幺正性破缺的问题，我们阐释了引入非局部引力的必要性。保持幺正性要求我们考虑指数型和渐近多项式型非局部算符，这类算符的构造基于三大原则:(i) 低能下还原为广义相对论，(ii) 理论传播子在紫外区满足预期行为，从而实现理论的可重整性甚至有限性，(iii) 有望解决爱因斯坦理论的经典奇点问题。尽管非局部性的引入会因给初始条件问题的求解带来困扰而被认为存在缺陷，但扩散方法为澄清这些表观问题提供了严谨的解决方案。

We have explored the way one quantizes the nonlocal gravitational theory following standard techniques of perturbative field theory. The nonlocal components in the Lagrangian improve the convergence of the Feynman diagrams and avoid the introduction of new poles in the graviton propagator, so that unitarity is preserved. Furthermore, we have shown that this theory exhibits asymptotic freedom, a property that allows one to neglect graviton interactions at extremely small distances. In particular, this property could lead to the avoidance of singularities in the classical regime.

我们已经探究了按照微扰场论的标准技术对非局部引力理论进行量子化的方法。拉格朗日量中的非局部分量改善了费曼图的收敛性，且不会在引力子传播子中引入新极点，因此得以保持幺正性。此外我们还证明，该理论具有渐近自由性质——这一性质使得极高极小尺度下引力子的相互作用可以忽略，尤其还可能帮助经典领域避免奇点的产生。

In the near future, it would be useful to extend this analysis including matter content in the theory, since it is believed that all the astronomical black holes in the universe have been formed by a gravitational collapse. In this way, one would be able to characterize the matter content in the classical singularity and check if this singularity would be smoothened out by means of nonlocality. It should not be disregarded that, as explained in section ”Nonlocal Classical Gravity”, the equations of motion derived from this theory are extremely complicated and, in general, exact solutions will be difficult to find unless we consider Ricci-flat spacetimes. A new formulation of NLQG based on a non-minimal coupling between gravity and matter fields overcomes these issues easily [88-90].

未来近期，在分析中纳入物质内容将会是很有意义的拓展，因为目前认为宇宙中所有天文黑洞都是由引力坍缩形成的。通过这种拓展，我们就可以刻画经典奇点中的物质组分，检验奇点是否能借助非局部性被抹平。不容忽视的是，正如“非局部经典引力”一节所述，该理论推导出的运动方程极为复杂，一般情况下很难得到精确解，只有里奇平直时空是例外。而基于引力与物质场非最小耦合的新非局部量子引力表述可以轻松解决这些问题 [88-90]。

Whereas this nonlocal approach provides engaging answers at the theoretical level, quantum effects are expected to manifest at Planck energies, so that the experimental evidence of these theories will be difficult to measure. Nevertheless, the rich properties of the theory may be used to apply the nonlocal formalism to other branches of physics, such as condensed matter, particle physics and cosmology, where the energy scale of the processes are far below the Plank scale. Recent results in the new NLQG formulation with non-minimal matter coupling points towards observable consequences in the cosmology of gravitational waves [91].

尽管这种非局部方法在理论层面给出了引人关注的结论，但量子效应预计只会在普朗克能量下显现，因此很难通过实验观测获得这些理论的证据。不过，该理论丰富的性质可以让非局部形式体系应用于物理学的其他分支，例如凝聚态物理、粒子物理和宇宙学，这些领域研究过程的能标远低于普朗克能标。采用非最小物质耦合的新非局部量子引力表述的最新研究表明，该理论在引力波宇宙学中存在可观测效应 [91]。

In conclusion, we have reviewed the nonlocal formulation of quantum gravity with exponential and asymptotically polynomial operators that succeeds in building a renormalizable and unitarity, or even completely UV-finite theory. Moreover, the use of nonlocal operators can provide a reasonable solution to the long-standing singularity problem. Although the empirical check of this class of theories may not be available yet, it could be imminent [91] and their consistency and robustness signal a promising scope of applicability, possibly in other branches of physics other than quantum gravity at the Planck scale.

总而言之，我们回顾了采用指数型和渐近多项式算符的非局部量子引力表述，该表述成功构建了一个可重整、保幺正，甚至在紫外区完全有限的理论。此外，非局部算符的使用可以为存在已久的奇点问题提供合理的解决方案。尽管这类理论的经验验证目前仍无法实现，但它可能近在眼前 [91]，而这类理论的自洽性与稳健性预示了其广阔的应用前景，除普朗克尺度的量子引力外，还可能应用于物理学的其他分支。

Cross-References

交叉引用

Mathematical Aspects of Analytic Infinite Derivative Gravity Theories

解析无穷导数引力理论的数学层面

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